开拓创新，激发卓越潜能：
以数学荣誉教育为例探索拔尖创新人才培养模式

教学成果应用及效果证明材料目录

附件1

上海市教育委员会文件

沪教委高〔2020〕6号

上海市教育委员会关于转发《教育部办公厅关于公布2019年度国家级和省级一流本科专业建设点名单的通知》的通知

各市属本科高等学校:

现将《教育部办公厅关于公布2019年度国家级和省级一流本科专业建设点名单的通知》(教高厅函〔2019〕46号)转发给你们。请各校按照文件精神，根据一流专业建设的规划和要求，持续提升专业内涵和建设水平，发挥一流专业的示范辐射作用。

附件:1.教育部办公厅关于公布2019年度国家级和省级一流本科专业建设点名单的通知

2. 上海市属高校2019年度国家级和省级一流本科专业建设点名单（分校下达）

上海市教育委员会

2020年1月16日

上海市教育委员会办公室 2020年1月17日印发

2019年度国家级和省级一流本科专业建设点名单

学校名称：上海纽约大学

序号	专业名称	国家级/省级	备注
1	数学与应用数学	省级	

附件2

教育部举行“教育这十年”“1+1”系列发布会（第十三场）

国务院新闻办公室网站 www.scio.gov.cn 2022-09-20 来源：教育部网站

原标题：“教育这十年”“1+1”系列发布会：介绍党的十八大以来教育国际合作交流情况

9月20日，教育部举行新闻发布会，请教育部国际合作与交流司司长刘锦，中国联合国教科文组织全国委员会秘书长秦昌威、北京师范大学国际与比较教育研究院院长刘宝存、国际高等教育创新中心（联合国教科文组织二类中心）主任李铭、上海市教育委员会主任王平、上海纽约大学校长童世骏介绍有关情况，并回答记者提问。教育部新闻办主任、新闻发言人续梅主持发布会。

新闻发布会现场

教育部新闻办主任、新闻发言人续梅：

各位嘉宾，各位记者朋友，大家上午好！今天在这里举行的是“教育这十年”“1+1”系列发布采访活动的第13场新闻发布会，欢迎各位的参加。今天的发布会主要向各位介绍党的十八大以来教育国际合作交流的有关情况。

党的十八大以来，以习近平同志为核心的党中央高度重视教育对外开放。在党中央、国务院的坚强领导下，教育国际合作交流水平全面提升，中国教育以更加开放自信主动的姿态走向世界舞台。今天我们特地邀请了几位嘉宾来为各位作介绍，主会场参加发布会的是，教育部国际合作与交流司刘锦司长，中国联合国教科文组织全国委员会秦昌威秘书长，本场专家是来自北京师范大学国际与比较教育研究院的刘宝存院长。

此次“1+1”活动的分会场设在了上海，参加发布会的是上海市教委王平主任，还有上海纽约大学童世骏校长。此外，国际高等教育创新中心的李铭主任在线参加今天的发布会。国际高等教育创新中心是我国与教科文组织合作设立的二类中心。

下面，首先请各位观看一个视频短片，一起了解一下党的十八大以来教育国际合作交流的有关情况。

（播放视频）

下面，我们首先请刘锦司长介绍教育国际合作交流的总体情况。

教育部国际合作与交流司司长刘锦：

各位媒体朋友，大家上午好！感谢大家参加“教育这十年”系列新闻发布会国际合作与交流专场。通过刚才的短片，大家可以看到近年来特别是这十年来教育对外开放、教育国际交流合作的做法和成效。感谢大家长期以来对教育国际合作与交流的关心支持。

党中央、国务院高度重视教育对外开放。习近平总书记指出：“推进教育现代化，要坚持对外开放不动摇，加强同世界各国的互容、互鉴、互通”。党的十八大以来，习近平总书记在一系列国际国内重大场合宣示扩大教育对外开放，多次作出重要指示批示，饱含深情给海外学子、留学归国人员、在华外国留学生、外国中小学生回信，为教育对外开放指明了方向，提供了根本遵循。中办、国办印发关于教育对外开放、中外人文交流的指导意见，国际合作与交流在我国教育事业中的地位和作用进一步凸显。

十年来，我们坚决落实党中央、国务院决策部署，更加注重开放的系统性整体性协同性，召开全国教育外事工作会议，印发《教育部等八部门关于加快和扩大新时代教育对外开放的意见》，推动中国教育以更加开放自信主动的姿态走向世界舞台。具体情况归纳为以下五个方面。

一是开放总体布局不断优化，教育的“朋友圈”更大了。我国同181个建交国普遍开展了教育合作与交流，与159个国家和地区合作举办了孔子学院（孔子课堂），与58个国家和地区签署了学历学位互认协议。深入实施共建“一带一路”教育行动，加强同共建国家教育领域互联互通，建设了23个鲁班工坊，启动了海外中国学校建设试

点。落实习近平主席重要倡议，成立“中国—东盟职业教育联合会”，设立中国上海合作组织经贸学院，启动“未来非洲—中非职业教育合作计划”，深化中国—中东欧教育交流合作，点面结合的区域教育合作机制不断完善。

二是开放高地建设不断提速，服务高质量发展的能力更强了。主动融入新发展格局，加快推进全方位开放。支持粤港澳大湾区建设国际教育示范区，支持长三角地区打造国际合作教育样板区和国际人文交流汇聚地，支持海南自贸港建设国际教育创新岛，与北京市合作设立“留学人才回国服务示范区”，助力京津冀一体化发展。与此同时，支持中西部和东北地区立足区位优势，扩大面向周边国家的教育开放。

引导高校通过国际合作与交流推进“双一流”建设，依托国家公派留学助力高校教师队伍建设和国际化人才培养，支持组建国际高校联盟，参与国际学术组织，推进跨学科交叉融合和跨领域、跨国界的科研合作。教育部于2018年启动国际产学研用合作会议以来，累计吸引70多个国家超过1.4万名专家学者参会，开展部门间和专家“一对一”科研合作2300多项，中外导师联合培养研究生4000多人。

三是改革促开放力度不断加大，内生源动力更足了。深化“放管服”改革，以信息化手段支撑全链条留学服务体系，开通“国家留学人才回国就业服务平台”。党的十八大以来，我国各类出国留学人员中超过八成完成学业后选择回国发展。与此同时，中外合作办学蓬勃开展，审批、管理、评估、退出机制不断完善。

过去10年，新增本科以上中外合作办学机构和项目中，理工农医类占比达65%。近年来，中外合作办学为缓解疫情导致的出国留学受阻发挥了积极作用，累计录取近万人。来华留学在推进制度建设、实施质量保障、严格入学标准、规范培养管理、加强留华毕业生工作等方面出台了一系列政策举措。2020—2021学年，在册国际学生来自195个国家和地区，学历生占比达76%，比2012年提高了35个百分点。

四是人文交流格局不断完善，中外“心联通”更紧了。党的十八大以来，中外人文交流形成了元首外交引领、领导人高访带动、高级别机制示范、双边多边结合、国内国外统筹、中央地方联动、官方民间并举的多元互动新格局。过去10年，教育部共举办中外高级别人文交流机制会议37场，签署300多项合作协议，达成近3000项具体合作成果。在人文交流机制框架下，形成了中美青年创客大赛、中俄同类大学联盟、中英中法百校交流、中南（非）职业教育联盟等教育品牌项目，为双边关系发展注入了正能量和暖力量。

五是参与全球教育治理不断深化，国际影响力更大了。围绕教育减贫、抗击疫情等全球性议题，我国持续加强与有关国际组织合作，共同实施了农村义务教育全面普及和质量提升、新冠肺炎疫情“安全返校行动”等项目。全面参与联合国教科文组织、二十国集团、金砖国家、亚太经合组织（APEC）、上海合作组织等多边机制框架下的教育合作，落实习近平主席关于成立“金砖国家职业教育联盟”、举办金砖国家职业教育技能大赛的倡议，主办金砖国家教

育部长会议。成功举办世界职业教育发展大会，积极筹办世界数字教育大会，搭建全球性高端教育合作平台，为全球教育治理贡献智慧和力量。

当前和今后一个时期，百年变局加速演进，开放合作成为推动新时代教育变革创新的关键要素，蕴含着新的发展机遇。我们深刻认识到，开放合作是建设高质量教育体系的内在要求，是办好人民满意教育的应有之义。面向未来，我们将坚持以习近平新时代中国特色社会主义思想为指导，自觉把党的全面领导、立德树人根本任务、社会主义办学方向落实到教育对外开放各领域全过程，扎根中国大地办教育，立足中国国情促开放。

我们将统筹国内国际两个大局，推进高水平制度型开放，畅通国内国际教育循环，构建面向全球的教育伙伴关系，全面提升中国教育的世界影响力。我们将在统筹发展和安全的前提下加强教育国际合作与交流，完善教育外事管理体系，确保新时代教育对外开放行稳致远。

谢谢。

续梅：

感谢刘锦司长，接下来我们请秦昌威秘书长介绍中国与联合国教科文组织合作的有关情况。

中国联合国教科文组织全国委员会秘书长秦昌威：

各位媒体界的朋友，大家好！感谢大家长期以来对我国与联合国教科文组织合作的关注。特别是过去十年，我与教科文组织合作在华举办的所有重大活动，都得到了媒体朋友的关心支持，一并表示感谢。党的十八大以来，在以习近平同志为核心的党中央坚强领导下，我国与教科文组织合作开启新篇章，深入参与全球人文治理迈上新台阶，服务国内教育、科学、文化、信息传播等领域改革发展取得新进展。

第一，党中央高度重视，领导人亲自推动，为我国与联合国教科文组织合作擘画崭新图景

习近平主席访问联合国教科文组织总部并发表重要演讲，多次会见教科文组织总干事并向双方合作举办的重大活动致贺信，为我国与教科文组织的合作指明方向。构建人类命运共同体和文明交流互鉴等重要思想理念在教科文组织平台日益得到广泛认同。彭丽媛教授应邀担任教科文组织促进女童和妇女教育特使，多次出席教科文组织女童和妇女教育奖主要活动，访问考察学校，接受教科文组织《信使》杂志专访，积极推动世界女童和妇女教育事业发展。中央领导同志多次出席我国与教科文组织合作举办的重大活动并发表讲话，分享中国成就、中国经验和中国方案，受到国际社会广泛关注和高度评价。

第二，合作举办多场高级别国际会议，为世界教育、科学、文化、信息传播事业发展注入生机活力

十八大以来，我国与教科文组织合作举办了“文化：可持续发展的关键”国际会议、首届国际学习型城市大会、世界语言教育大会、国际教育信息化大会、国际博物馆高级别论坛、国际职业技术教育大会、世界语言资源保护大会、国际人工智能与教育大会、第44届世界遗产大会等重要会议，围绕国际热点和前沿议题，搭建了国际交流与合作平台，形成了一批重要成果文件，成为教科文组织的重要文献，促进了国际社会在相关领域的合作发展。

第三，积极提供国际公共产品，为发展中国家提供切实支持

十八大以来，我国与教科文组织合作设立了女童和妇女教育奖，与之前设立的孔子教育奖、亚太地区教育创新文晖奖等奖项国际影响日益扩大，产生了良好的示范激励效应。10年来，共有来自24个国家的30个项目获得孔子教育奖，来自13个国家的13个项目获得女童和妇女教育奖，来自13个国家的16个项目获得文晖奖。我国支持设立的长城奖学金项目规模进一步扩大，10年间共资助来自93个国家的512名学生来华学习。中国—教科文组织信托基金项目惠及12个非洲国家，受到项目国欢迎。我国支持教科文组织旗舰杂志《信使》复刊，得到国际社会良好反响。合作设立的丝路青年学者资助信托基金项目得到各国青年学者的热烈响应和积极参与。疫情发生后，我国与教科文组织等国际组织合作编写《教育应对疫情参考手册》，为提高全球教育应急响应能力作出贡献。

第四，深入参与教科文组织重大倡议、重要议程、重点计划，为全球人文治理做出中国贡献

十八大以来，我国积极参与全球2030年可持续发展教育议程实施，深度参与《承认高等教育相关资历全球公约》等一系列重要国际规则制定工作。我国代表和专家学者在教科文组织相关机构和机制担任重要职务。

我国积极参与教科文组织重点计划和旗舰项目，10年来，共有15项世界遗产列入《世界遗产名录》，6个项目列入非物质文化遗产相关名录、名册，6项文献遗产列入《世界记忆名录》，15处地质公园加入教科文组织世界地质公园网络，5个生物圈保护区加入世界生物圈保护区。多个城市、项目及个人获得教科文组织教育、科技奖项。北京、杭州、成都、上海等4个城市获学习型城市奖，国家开放大学“一村一名大学生”项目获教育信息化奖，屠呦呦、李兰娟获国际生命科学研究奖，谢毅、陈化兰、张弥曼、胡海岚等获杰出女科学家奖。

我国与教科文组织设立多个合作平台，合作格局更加完善。10年来，双方合作在华设立了10个教科文组织二类中心和12个教席。地方政府与教科文组织合作力度不断提升，13个中国城市入选教科文组织创意城市网络，10个城市加入全球学习型城市网络。

下一步，我们将坚持以习近平新时代中国特色社会主义思想为指导，高举和平、发展、合作、共赢的旗帜，坚持共商共建共享原则，进一步全面深化与联合国教科文组织合作，共同造福世界各国人民，特别是支持发展中国家教育、科技、文化事业进步，为推动构建人类命运共同体做出新贡献。

谢谢大家！

续梅：

感谢秦昌威秘书长，接下来我们请分会场的王平主任介绍上海教育对外开放的有关情况。

上海市教育委员会主任王平：

主持人，各位嘉宾，媒体朋友们，大家上午好！党的十八大以来，上海深入贯彻习近平总书记关于教育的重要论述，全面落实习近平总书记关于教育对外开放的重要指示批示，着眼服务国家外交大局、服务教育强国建设、服务区域发展需要，以扩大教育对外开放推动改革、驱动创新、带动发展，发挥了全国教育改革试验田作用，推动了教育发展水平迈上新台阶，已总体达成《中国教育现代化2035》确定的教育事业发展和人力资源开发主要指标，教育服务力、贡献力和国际影响力显著提升。

第一，聚焦培养中国情怀、国际视野的优秀人才。我们牢记教育初心，用好优质国际教育资源培养一流人才。一是对标发展世界水平的基础教育。连续参加OECD组织的旨在测评学生学业质量的PISA项目和测评教师专业发展素养的TALIS调查，测评结果一直位居全球前列，为基础教育课程改革提供思路和借鉴，也向世界展示了我国基础教育的高水平。

二是集聚优质国际教育资源。坚持“以我为主、填补空缺、补弱增强”，举办高水平中外合作办学机构和项目176个，其中依托上海15所“双一流”高校举办的机构和项目数超过50%，包括：开办第一所中美合作独立法人的上海纽约大学，聚焦神经科学、应用数学、经济学等培养高层次人才；开办中加合作上海温哥华电影学院、中瑞合作上海洛桑酒店管理学院、中英合作上海国际时尚创意学院、中以合作上海交通大学国际农业与生态学院等，分类培养了影视全产业链人才、高端酒店管理人才和时尚文创人才。

三是促进青少年学生跨文化交流。每年资助2%在校学生赴海外学习实习，每年资助100名在校学生赴国际组织实习；构建非通用语种大中小学一体化培养机制，布点开展9种非通用语种教学，培养紧缺外事后备人才。每年开展国际友好城市青少年夏令营，探索实施“游学上海”计划，促进了跨文化交流。

第二，聚焦对接国家和区域对外开放战略所需。我们发挥地处改革开放前沿和“一带一路”建设桥头堡的区位优势，从三个方面积极作为：一是主动服务“一带一路”建设。扩招共建国家国际学生，上海高校学历留学生中68%来自沿线国家。鼓励高校与沿线国家实施“小而美”合作办学项目，建设了“一带一路”中老铁路工程国际联合实验室，面向老挝学生培养中老铁路技术更新和运维服务专门人才。依托上合组织国际司法交流合作培训基地等平台，针

对沿线国家2000余名高级官员开办研修班，培养了一批知华友华人士；依托WTO亚太培训中心，面向沿线国家开展多边贸易规则与治理培训。

二是依托自贸区建设深化教育服务业开放。布局扩大外籍人员子女学校办学规模，优化了营商环境；完善优秀留学生永久居留、在沪创新创业等机制，实施更加综合便利的境外人士融入服务措施，提升了城市吸引力。三是深化推进高校创新能力开放合作。16所高校获500余项国家自然科学基金国际合作项目，发表国际合作论文6万余篇。7所上海高校与国际顶尖大学在生物、工程、通信等领域组建9个国际合作联合实验室，5所上海高校牵头组织或参与国际大科学计划。举办国际青年学者论坛，实施“超级博士后”计划，吸引了一批境外优秀青年学者来沪工作，超过80%来自海外名校。

第三，聚焦提升社会主义现代化国际大都市教育影响力。我们围绕建设“亚洲最受欢迎留学目的地城市之一和富有活力和吸引力的国际教育交流中心城市”，推动上海成为全球人才向往之地。一是强化在沪国际学生融合教育。全市在读国际学生近8万人，上海高校留学生中学位生占65%。构建起国际学生招生入学、预科教育、英语授课和社会服务支撑体系，建立国际学生中国文化体验和实践基地。

二是勇担我国教育“走出去”排头兵。实施中英数学教师交流项目，推广上海数学教学模式，英文版《真正的上海数学》赢得“世界领先的数学教程”美誉。9所高校在14个国家开展海外办学，助力中企“走出去”；上海职业院校积极在境外建设“毕昇工坊”等基地，推动我国技术和标准对外辐射。作为全球“可持续发展教育”专题组协调城市，协同全球100余个城市开展实践探索，加入“全球学习型城市网络”，获评“联合国教科文组织学习型城市”。

三是积极参与贡献全球教育治理。合作建立联合国教科文组织教师教育中心，实现联合国二类机构在沪“零的突破”。引驻国际戏剧协会总部，开创国际文化组织总部迁址亚洲的先例。支持上海开放大学持续担任联合国教科文组织“远程与开放学习教席/姊妹大学网络（东亚）”主持单位，推动开放远程教育领域的知识传播、资源共享和学术发展。设立联合国国际海事组织亚洲海事技术合作中心，为制定国际海事规则提供中国方案。创建中医药国际标准化研究中心，打造全球传统医学总部，为我国参与全球治理贡献上海智慧。

新时代新征程，上海将紧紧围绕服务国家发展战略，实施更深层次、更高质量、更加安全的教育对外开放，助力发展同具有世界影响力的社会主义现代化国际大都市相匹配的一流教育。

谢谢。

续梅：

感谢王平主任，最后请刘宝存院长对十年来教育国际合作交流以及中国与联合国教科文组织合作相关工作进行点评。

北京师范大学国际与比较教育研究院院长刘宝存：

各位媒体朋友，大家上午好。刚才刘锦司长、秦昌威秘书长详细介绍了我国十年来教育国际合作交流取得的成绩。正如两位所讲的那样，党的十八大以来，我国坚持对外开放不动摇，不断开创教育对外开放新格局，全面提升教育国际合作交流水平，形成了更全方位、更宽领域、更多层次、更加主动的教育对外开放局面，增强了中国教育的国际影响力和亲和力。

第一，我国坚持以顶层设计为先导，不断完善教育对外开放政策，这是我国教育对外开放取得丰硕成果的基础。2016年，中共中央办公厅国务院办公厅印发《关于做好新时期教育对外开放工作的若干意见》，要求坚持扩大开放，做强中国教育。教育部印发《推进共建“一带一路”教育行动》的通知，聚力构建“一带一路”教育共同体，形成平等、包容、互惠、活跃的教育合作态势，促进区域教育发展，全面支撑共建“一带一路”。2020年，教育部等八部门印发《关于加快和扩大新时代教育对外开放的意见》。正是在各项政策的引领和推动下，我国不断完善教育对外开放总体布局，统筹规划、重点推进，逐步形成了更全方位、更宽领域、更多层次、更加主动的教育对外开放局面。

第二，我国坚持以“一带一路”教育行动为支点，构建教育对外开放新格局，这决定了我国教育对外开放的战略格局。“坚持推动构建人类命运共同体”是习近平外交思想的集中体现。我国以“人类命运共同体”思想为指导，全面推进共建“一带一路”教育行动，积极开展共建国家双边多边人文交流高层磋商，加强规划对接和政策磋商，商定“一带一路”教育合作交流总体布局，探索教育合作交流的新模式，增进教育合作与交流的广度和深度，协调推动共建各国建立教育双边多边合作机制，与共建国家形成了平等、包容、互惠、活跃的教育合作态势。“一带一路”教育行动的实施，进一步推动了我国教育的全面对外开放。

第三，我国坚持以人才培养为核心，全面推动教育对外开放，这决定了我国教育对外开放的重心和成效。教育的本质属性是培养人才，教育对外开放虽然在某种意义上超越了狭义的人才培养，但根本目的是服务于人才培养。

党的十八大以来，我国积极推动同其他国家学历学位互认、标准互通、经验互鉴；积极发展出国留学教育，充分利用国际优质教育资源培养我国现代化建设急需的国际化人才；实施“留学中国”计划，建立并完善来华留学教育质量保障机制，全面提升来华留学质量；积极推动教育的在地国际化，引进国际优质教育资源，开展中外合作办学，丰富我国教育供给，创新人才培养模式，推动国际化人才培养；推进中外人文交流特别是高级别人文交流机制建设，促进教师、学生流动和校际交流，加快培养具有国际视野和全球竞争力的人才。

第四，我国坚持以加强与联合国教科文组织等国际组织和多边机制的合作为抓手，积极参与全球教育治理，这决定了我国教育对外开放的国际情怀。党的十八大以来，我国深入参与联合国教科文组织的重大倡议、重要议程、重点计划；积极提供国际公共产品，为发展中国家提供切实支持；与联合国教科文组织合作举办首届国际学习型城市大会、世界语言大会、国际教育信息化大会、国际人工智能与教育大会、世界职业技术教育发展大会等高级别国际会议；全面参与联合国教科文组织、二十国集团、金砖国家、亚太经合组织（APEC）、上海合作组织等多边机制框架下的教育合作，为全球性教育问题的解决、创建基于人类命运共同体理念的全球教育治理机制，贡献了中国智慧、中国方案、中国力量。

回顾过去这十年，我国教育对外开放取得的成就是有目共睹的。着眼未来，我国社会经济高质量发展的要求与世界格局的大变革和不确定性的增加需要我们以更大的格局、更高的眼光、更多的智慧去开创教育对外开放的新局面，相信我们在习近平新时代中国特色社会主义思想的指导下，坚持对外开放不动摇，敢于担当，勇于创新，善于谋划，勤于实干，一定可以交出一份无愧于时代、无愧于人民的精彩答卷。

谢谢大家！

续梅：

感谢刘宝存院长，我们几位嘉宾的介绍就到这里，下面进入答问环节，看看记者朋友们有没有问题。

新华社记者：

我们知道，明年“一带一路”将进入第十个年头，请问教育部在推动“一带一路”教育高质量发展方面有什么新的举措？谢谢。

刘锦：

感谢提问。下面我简要向大家介绍推进共建“一带一路”教育行动高质量发展的一些考虑和举措。

习近平总书记提出“一带一路”倡议以来，教育部积极贯彻落实，印发了《共建“一带一路”教育行动》，我国与共建国家不断加强教育合作交流，同时也带动了地方和各级各类学校广泛参与到“一带一路”建设中来。

明年，“一带一路”建设将进入第十个年头。面向未来，我们将坚持以高质量发展为主题，以绘制“工笔画”为主线，以高标准、可持续、惠民生为目标，努力将“一带一路”建设成为我国的全球教育伙伴集聚区、国内国际教育

循环示范区、中国教育国际影响力辐射区，为我国建设高质量教育体系提供发展新空间。下一步，将主要从以下几方面重点发力。

一是抓住人才培养这个基础。加强基础学科和前沿核心技术领域人才培养，面向“一带一路”深化国际产学研用合作，支持高校引进世界一流教育资源。加强各类国际化人才培养，加大共建国家本土人才培养力度，做优做强来华留学，主动对接“一带一路”重大项目人才需求。

二是抓住互联互通这个关键。巩固拓展教育合作伙伴，提升教育互通互认水平，提高高等教育学历学位、职业技能等级证书互认水平，向共建国家共享更多优质数字教育资源。

三是抓住民心相通这个根本。引导广大师生树立人类命运共同体理念，推进中外青少年友好交流。继续发挥高校和职业院校在民心相通方面的独特作用，鼓励共建国家高校及科研机构合作开展文明互鉴研究，为各国人民相知相容相通提供理论支撑。

谢谢大家。

中国日报记者：

教育数字化变革已经成为全球教育变革的重要方向，刚刚结束的教育变革峰会也强调了数字的教与学，请问我国在教科文组织平台推动教育数字化开展了哪些工作？

秦昌威：

谢谢你的提问。

推动教育数字化变革既是教育适应未来的必然选择，也是全球实现2030年教育目标的重大机遇，刚刚记者朋友提到，昨天结束的联合国教育变革峰会也强调了数字化变革的重要意义，并发出了相关行动倡议。我国高度重视数字技术对教育的革命性影响，并取得显著成效，过去十年来与联合国教科文组织开展了一系列活动，是双方教育合作的重点、亮点，归纳起来有三个方面：

一是主动发起，共商教育数字化变革大计

2015年，我国与联合国教科文组织合作举办国际教育信息化大会，习近平主席向大会致贺信，指出要推动教育变革和创新，构建网络化、数字化、个性化、终身化的教育体系，得到国际社会高度认同。会议通过的成果文件《青岛宣言》，成为本领域标志性文件，为全球教育信息化发展提供了政策建议和行动指南。

又如2019年，我国与教科文组织共同举办国际人工智能与教育大会，这是我国主动发起、全球该领域首次国际会议，与会代表认为具有重要里程碑意义。习近平主席在贺信中提出“要加快发展伴随每个人一生的教育、平等面向每个人的教育、适合每个人的教育、更加开放灵活的教育”，会议通过重要成果文件《北京共识》，成为智能时代全球教育治理的重要文献。其后，我国与教科文组织已连续两年合作举办该会议，今年还将继续举办该会议，已成为该领域的国际品牌会议。

二是积极倡导，为教育数字化变革提供中国智慧

2021年，联合国教科文组织促进女童和妇女教育特使彭丽媛教授在出席我国与教科文组织合作设立的女童和妇女教育奖颁奖仪式时强调，应全面推动女童和妇女数字教育，充分运用数字技术，帮助她们在数字时代拥有幸福生活。

我国深入参与教育变革峰会筹备工作，积极倡导教育数字化变革。怀进鹏部长担任2030年教育高级别指导委员会成员，在出席峰会预备会时分享了我国在教育数字化变革方面的做法和主张。我国还作为亚太地区代表担任峰会咨询委员会成员，积极参与有关主题行动领域磋商，并参与峰会成果文件起草编制，为形成教育数字化变革的全球共识与行动方向贡献了力量。

三是推动各方参与，形成推动教育数字化变革合力

我国高校、地方政府、企业等与教科文组织积极开展合作，共同推动教育数字化变革。比如，疫情发生后，我国与教科文组织、非洲国家召开三方合作抗疫会议，动员相关伙伴方共享数字教育资源，支持非洲国家；设在北京师范大学的教科文组织二类中心国际农村教育研究与培训中心组织多场国际会议，分享教育信息化应对疫情与教育脱贫攻坚的经验；设在深圳的国际高等教育创新中心发起成立国际网络教育学院，推动数字技术赋能发展中国家高校教师；北京、上海等城市依托教科文组织全球学习型城市网络，传播数字化支撑终身学习体系建设的经验；华为、伟东等一批企业与教科文组织合作开展教育数字化的项目，服务发展中国家数字技能培养和数字教育平台建设，都取得了良好成效。

下一步，我们将进一步加强与联合国教科文组织在数字教育领域的合作，携手推进教育数字化变革，为推动我国教育现代化和全球教育发展作出更大贡献。谢谢。

华夏时报记者：

我有一个问题想请问上海分会场的嘉宾。我们知道上海纽约大学是中外合作办学的代表，请问在加强中外大学合作、推进国际化人才培养上有哪些好的思路和举措？谢谢。

上海纽约大学校长童世骏：

谢谢这位媒体朋友。党的十八大以来，在教育部、上海市教委和浦东新区政府的高度关心和重视下，上海纽约大学应发展之需、答时代之问，以高水平开放、高质量发展的势头，为国际合作和人才培养搭建了交流窗口和实践平台。主要体现在以下几个方面。

第一，立足上海浦东，成为中国高等教育改革的“试验田”。上海纽约大学成立于2012 年，是第一所中美合办的研究型大学，也是纽约大学具有学位授予资格的三大校园之一。学校于2013年迎来首届本科生，2014年入住陆家嘴世纪大道校园。在建校10周年之际，即将搬入作为永久校址的前滩校区。目前在校本科生约2000人，中外学生比例1:1；师生比保持在1:8以内。学科专业设置围绕中国发展、上海城市创新、浦东开发开放所需的国际化、创新型人才布局，重点发展包括神经科学、数学、金融学、数据科学等在内的新兴、交叉学科。

第二，“引进来，走出去”，搭建起中美人文交流的桥梁。上海纽约大学是中国高等教育开放、自信的一张名片，为增进两国青年之间的友谊贡献了宝贵的力量。学校常务副校长杰夫•雷蒙被评选为40位“改革开放40周年最具影响力的外国专家”之一，获得了“上海市荣誉市民”称号。疫情以来，学校充分发挥全球学术资源配置的优势，一方面为缓解出国留学生境外求学困难提供有力支援，另一方面也在政府及社会各界的支持下为国内外师生回沪返校提供支持与保障。去年秋天新生开学之际，中国驻美国大使秦刚也以视频的方式为上海纽约大学学子送上祝福，彰显了中美高校互学互鉴、“引进来，走出去”的韧性。

第三，发挥合办大学平台优势，深化国内外学术合作。通过共享学术资源、建设联合研究中心、合作培养研究生等形式，学校实现了与两所母体大学，华东师范大学和纽约大学的协同共进。学校在多个优势领域建立了科研平台，包括6所联合研究中心、城市设计与城市科学重点实验室、金融波动研究所等。实力雄厚的科研平台吸引了一批海内外顶尖人才的加盟，包括两位诺贝尔经济学奖得主——罗伯特•恩格尔和托马斯•萨金特。

人才培养是高校中心任务。学校始终将建设世界一流大学，服务创新经济和人类命运共同体作为办学重点。坚持立德树人，定位于培养兼具家国情怀和全球视野的国际化创新型人才，走特色发展道路。

为培养学生的“创新能力”和“全球胜任力”，学校以博雅教育的理念为基础，为学生精心设计了一套通识培养与专业训练相结合的课程体系，在为学生打下扎实专业基础的同时，充分拓宽学术视野，训练审辩思维能力，培育健全的人格。

与此同时，学校秉承平等、融合的原则，以规范化、一体化的管理和国际化、高质量的教育品质，保障国际学生的人才培养质量，打造来华留学新品牌，努力为把中国建成具有全球影响力的留学目的国贡献力量。

谢谢大家。

澎湃新闻记者：

我们刚刚有介绍到，国际高等教育创新中心是我国与联合国教科文组织合作设立的二类中心，我想请问，在推动与发展中国家加强高等教育合作上主要做了哪些工作？谢谢。

国际高等教育创新中心（联合国教科文组织二类中心）主任李铭：

感谢记者所提的这个问题。联合国教科文组织高等教育创新中心是于2015年经联合国教科文组织第38届大会批准，由教科文组织和深圳市政府共同设立的教科文组织二类机构，永久办公地点设置于南方科技大学。

创新中心以“赋能高校教师的数字化教学能力”为抓手，关注发展中国家高校及教师的数字化转型，并通过一系列的培训项目，进行了促进国际高等教育数字化协同发展的多种有益尝试。

中心参与了由深圳市政府出资200万美元、与教科文组织总部合作设立的“教科文组织—深圳信托基金”项目实施，帮助了非洲十个国家建立或提升高等教育质量保障体系，在柬埔寨、斯里兰卡两个亚洲国家开展了数字化教学能力的培训，成为了教科文组织高教领域一个成功的项目。

创新中心联合中国伟东云教育、创显科教、希沃等教育科技企业，面向亚非34个发展中国家高校捐献教育装备，建设智慧教室。已经落成的多间智慧教室，成为了发展中国家教师开展数字化教学培训的重要平台。

2019年，创新中心联合全球15所发展中国家院校、9家信息科技企业共同发起设立了国际网络教育学院（IIOE），在全球疫情蔓延的背景下，帮助发展中国家的高校教师适应在线教学，发挥了重要的作用。目前IIOE已经与全球30多个国家的高校建立了合作关系，服务了135个国家超过1万名教师。IIOE秉承“共商、共建、共享”的原则，正致力打造涵盖国际组织、发展中国家高校和教育科技企业的高教数字化转型联盟。

创新中心还致力于高等教育数字化转型研究，与清华大学教育研究院合作、携手全球50多位教育专家学者推出了《混合教学改革手册》《高等教育教师发展手册》《职业教育教师发展手册》 和《高等教育教学数字化转型研究报告》。这一系列研究成果在第三届世界高等教育大会上正式向全球发布，并将持续为IIOE等项目的发展贡献学术智慧。

过去十年间，我们见证了中国高等教育快速发展、不断变革。面向未来，创新中心将继续成为全球高等教育数字化转型的推动者，为促进联合国可持续发展目标4的实现和后疫情时代的全球教育重建继续做出我们的贡献。

我的回答完毕，谢谢各位媒体朋友们。

续梅：

感谢李铭主任。如果记者朋友们没有什么问题，我们第13场“教育这十年”新闻发布会就开到这里，请各位记者朋友稍事休息，我们继续举行第14场“教育这十年”的新闻发布会。

【纠错】［责任编辑：司马屹杰］

相关链接>>

- 教育部举行“教育这十年” “1+1”系列发布会（第十二场）
- 国务院学位委员会办公室就新版研究生教育学科专业目录和目录管理办法答问
- 教育部就新版《职业教育专业简介》答问
- 教育部举行“教育这十年” “1+1”系列发布会（第十一场）
- 教育部举行“教育这十年” “1+1”系列发布会（第十场）

上海在读国际学生近8万人 申城:全球人才向往地

2022年09月21日16:58 | 来源：解放网

上海在读国际学生近8万人

申城：全球人才向往地

本报讯（记者 徐瑞哲）“围绕建设‘亚洲最受欢迎留学目的地城市之一和富有活力和吸引力的国际教育交流中心城市’，申城正在成为世界各国人才向往之地。”在教育部昨天举行的“教育这十年”“1+1”第十三场新闻发布会上，上海市教委主任王平介绍说，上海全市在读国际学生有近8万人，其中学位生占比65%。

王平表示，上海积极参与全球教育治理，包括合作建立联合国教科文组织教师教育中心，实现联合国二类机构在沪“零的突破”；引驻国际戏剧协会总部，开创国际文化组织总部迁址亚洲的先例；支持上海开放大学持续担任联合国教科文组织“远程与开放学习教席/姊妹大学网络（东亚）”主持单位，推动开放远程教育领域的知识传播、资源共享和学术发展；设立联合国国际海

返回顶部

事组织亚洲海事技术合作中心，为制定国际海事规则提供中国方案；创建中医药国际标准化研究中心，打造全球传统医学总部，为我国参与全球治理贡献上海智慧。

上海举办高水平中外合作办学机构和项目达176个，其中依托上海15所“双一流”高校举办的机构和项目数超过50%。譬如开办第一所中美合作独立法人的上海纽约大学，聚焦神经科学、应用数学、经济学等培养高层次人才；开办中加合作上海温哥华电影学院、中瑞合作上海洛桑酒店管理学院、中英合作上海国际时尚创意学院、中以合作上海交通大学国际农业与生态学院等，分类培养了影视全产业链人才、高端酒店管理人才和时尚文创人才。上海纽约大学校长童世骏透露，作为中国高等教育改革的“试验田”，上纽大在多个优势领域建立科研平台，包括6所联合研究中心、城市设计与城市科学重点实验室、金融波动研究所等。

上海还勇当我国教育“走出去”的排头兵，实施中英数学教师交流项目，推广上海数学教学模式，英文版《真正的上海数学》赢得“世界领先的数学教程”美誉；9所高校在14个国家开展海外办学，助力中企“走出去”。作为全球“可持续发展教育”专题组协调城市，上海协同全球100余个城市开展实践探索，获评“联合国教科文组织学习型城市”。

返回顶部

教育是面向未来的事业，上海每年资助2%在校学生赴海外学习、实习，每年资助100名在校学生赴国际组织实习。

（来源：解放日报）

(责编：严远、轩召强)

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上纽大高水平开放、高质量发展

成为中国高等教育改革的“试验田”

记者 魏小潭

“上海纽约大学是中国高等教育开放、自信的一张名片。”在近日教育部召开的“教育这十年”“1+1”系列发布会上，上海纽约大学校长童世骏在回答记者提问时表示，上海纽约大学应发展之需、答时代之问，以高水平开放、高质量发展的势头，为国际合作和人才培养搭建了交流窗口和实践平台。“立足上海浦东，学校成为中国高等教育改革的‘试验田’。”

探索新模式，成为中国高等教育改革的“试验田”

上海纽约大学成立于2012年10月，也是纽约大学具有学位授予资格的三大校园之一。在建校10周年之际，学校即将搬入作为永久校址的前滩校区。作为第一所中美合办的研究型大学，学校自诞生起就肩负着建设我国高等教育改革“试验田”的使命。上纽大定位于培养具有全球视野的国际化创新人才，将“创新”理念融入学校建设、管理和评价的各个方面。

截至今年，在校本科生约2000人，中外学生比例1:1；学校践行小班化教学，师生比长期保持在1:8以内，在全球高校中处于领先水平。

学校在师资引进和人才评价方面也借鉴世界一流大学的成功经验和做法，按照同等或高于纽约大学的学术标准，面向全球公开招聘优秀师资，并在多个跨学科研究的优势领域建立了相应的研究中心及联合研究中心，如城市设计与城市科学重点实验室和金融波动研究所等。实力雄厚的科研平台吸引了一批包括诺贝尔奖获得者、各国院士在内的海内外顶尖人才的加盟，其中就有两位诺贝尔经济学奖得主——Robert Engle和Thomas J. Sargent。

“引进来，走出去”，搭建起中美人文交流的桥梁

作为中美人文交流的桥梁和纽带，上海纽约大学为增进两国人民，尤其是青年间的友好交往贡献了宝贵的教育力量。

上海纽约大学常务副校长杰夫·雷蒙是一位行走在校园里的杰出“文化使者”。他曾被国家外国专家局评选为40位“改革开放40周年最具影响力的外国专家”之一，获得了“上海市荣誉市民”的称号，也在上海市举办的纪念中美乒乓外交50周年系列活动中拿起球拍，一展身手。此外，国际学生中涌现出许多与中国文化结下不解之缘的“中国通”，他们说中文、弹古筝、打快板……上海纽约大学学以致用的课堂和丰富多彩的社会实践，拓宽了国际学生实现“中国梦”的途径。

疫情期间，学校充分发挥在全球范围配置学术资源的办学优势，为疫情期间出国留学生境外求学困难提供有力支援。学校通过与纽约大学合作，2020~2021学年成功接受了一批因受疫情防控影响无法按时出国的中国籍学生，前来学校参加“就近就读”学分项目。

发挥合办大学平台优势，深化国内外学术合作

学校在本科和研究生人才培养中大范围地引进了纽约大学课程、师资、学术辅导等优质办学资源，与纽约大学多个院系实现课程互选和更深层次的学术资源共享，为进一步实现更多学科的人才培养、优质学术资源共享、联合科研平台共建等提供坚实保障，并取得了突出的成绩。

童世骏表示，人才培养是高校中心任务。学校始终将建设世界一流大学，服务创新经济和人类命运共同体作为办学重点。坚持立德树人，定位于培养兼具家国情怀和全球视野的国际化创新型人才，走特色发展道路。

童世骏介绍，上海纽约大学为培养学生的“创新能力”和“全球胜任力”，以博雅教育的理念为基础，为学生精心设计了一套通识培养与专业训练相结合的课程体系，在为学生打下扎实专业基础的同时，充分拓宽学术视野，训练审辩思维能力，培育健全的人格。

与此同时，学校秉承平等、融合的原则，以规范化、一体化的管理和国际化、高质量的教育品质，保障国际学生的人才培养质量，打造来华留学新品牌，努力为把中国建成具有全球影响力的留学目的国贡献力量。

奋进新征程 建功新时代

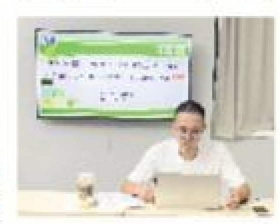
奉贤区金汇成校“喜迎二十大”专题党课开讲

为迎接党的二十大召开，激励党员坚定理想和信念，引导和激发广大党员团结在以习近平总书记为核心的中共中央周围，提高金汇镇党员干部为民服务的宗旨意识，金汇成校“喜迎二十大”专题党课开讲了。本次讲座主题为《奋进新征程 建功新时代》，由金汇成校校长主持，特聘教师封惠平主讲，来自金汇镇各村居委、中小幼学校等194家基层单位，共1000余名党员参与了本次讲座。

本次讲座以线上线下相结合的方式举行，突破空间局限，将党课讲台放入直播间，以金汇成校为主会场，各分会场通过微信扫码观看直播，让金汇镇各基层单位党员相约在云端，共同学习强化理论知识。讲座中，封惠平从党的二十大的重大意义，重大历史关头的重要讲话和我们有信心、有底气把自己的事情做好这三个方面展开。他强调，我们的信心是全方位的、是开创性的、也是根本性的，我们的底气来自于新时代10年的伟大变革，经济持续发展取得历史性成就，脱贫攻坚取得决定性胜利，人民民生福祉不断增进，企业技术创新能力建设成效显著。

课后，与会现场的金汇成校党员教师表示这是一堂内涵深远、意义深刻的专题党课，更进一步了解到中国共产党在中华民族伟大复兴过程中的重要作用，线上的党员同志也纷纷在评论区留言评论，感受到了作为一名党员的责任感、使命感、荣誉感，自己将为加强基层党组织建设做出一份贡献，做好党员模范带头作用，以实际行动践行为民服务宗旨。

三金一银：开大奉贤分校在第八届中国国际“互联网+”大学生创新创业大赛上海赛区中取得佳绩



近日，第八届中国国际“互联网+”大学生创新创业大赛上海赛区赛事工作圆满落幕，上海开放大学奉贤分校在比赛中取得佳绩。比赛期间，共有来自上海赛区职教赛道的105个项目入围市决赛，上海开大系统有8支队伍参加决赛，开大奉贤分校占4个名额。最终，开大奉贤分校学生同上海赛区职教赛道的学生激烈角逐，收获金奖三个、银奖一个。学校教师赵国辉、王煜炜、仇保妹、梁益智、周亚玲收获优秀指导教师奖。

本次市赛从2022年6月开始组织筹备到8月上旬结束，经历了校赛、市赛近3个月的层层选拔和竞赛。开大奉贤分校在市开大和区教育局的关心指导下，精心打磨参赛项目，师生协作，勇于拼搏，展现了开大人良好的创新创业精神。目前，第八届双创大赛国赛阶段已正式拉开序幕，开大奉贤分校“祥凡科技”项目将代表国家开放大学参与国赛，同全国职教赛道学生同台竞技，期待学校师生能再创佳绩！

中国国际“互联网+”大学生创新创业大赛旨在深化高等教育综合改革，激发大学生的创造力，促进“互联网+”新业态形成，以创新引领创业、创业带动就业，推动高校毕业生更高质量创业就业。开大奉贤分校将会继续加大培养力度，为学生提供更多学习渠道，鼓励学生积极响应国家“大众创新、万众创业”的号召，提升教师专业指导能力和水平，营造师生共创的教育氛围，进一步深化双创教育成果，全力推进开放教育的高质量建设，为打造“自然、活力、和润”的南上海品质教育区贡献力量，以实际行动迎接党的二十大胜利召开。

弘扬传统精神 传承武术文化

奉贤区洪庙小学武术队在第二届全国少儿武术大赛中斩获佳绩

第二届全国少儿武术锦标赛网络赛由中国武术协会主办，北京元海汇教育科技有限公司承办，湖南湘体科技有限公司协办，是目前最顶级权威的全国性少儿武术赛事。来自全国各地众多学校、幼儿园、社会武术机构、俱乐部的4000多名运动员报名参加了此次比赛。

奉贤区洪庙小学武术队的精武少年们在武术训练中，勤奋练习，吃苦耐劳，烈日下有他们挥汗如雨的身影，严寒中有他们矫健飒爽的英姿。一日日地刻苦训练培养了孩子们不畏险阻、坚持不懈的坚毅品质。队员们积极迎接本次武术大赛，他们跟随节奏，施展拳脚，刚柔并济的集体拳术套路动作展现了孩子们积极向上、阳光朝气的精神面貌。最终，洪庙小学“精武少年”武术队不负众望，取得优异成绩：荣获三个一等奖、五个二等奖、一个三等奖，学校自选集体项目荣获少儿乙组(7~9周岁)三等奖。

武术是中华民族的优秀瑰宝，一招一式、一拳一脚都展现着运动美和艺术美。练习武术不仅强身健体，更培养了学生良好的武德。愿奉贤区洪庙小学学子们继续以武强身，以武怡情，享受武术带来的成长与快乐。

嘉定区新成路小学：师徒携手，共话成长

为了加快青年教师队伍建设，充分发挥骨干教师“传、帮、带”的引领作用，9月16日下午，新成路小学组织召开“2022学年度拜师结对会”。通过拜师结对，促进青年教师在学科教学、班级管理上的提升，打造更优质的教师团队。

副校长姚海萍代表学校，宣布2022学年度新小拜师结对导师名单，同时感谢所有带教导师的热情参与。

校长袁亚萍为带教导师们颁发了学校聘书，师徒双方郑重签署“师徒结对协议书”，明确职责。相信在导师们的引领下，新小的青年教师能够迅速成长起来。

陈卫老师作为导师团队代表发言。他给新教师们分享了四句话：一是做实学习；二是做细教研；三是做优听课；四是做好准备。希望所有新教师都能做到：一年站稳讲台，三年站好讲台，五年成就讲台。吴蕾老师作为青年教师代表分享自己作为新教师、新班主任在新小团队中的开学工作经历。

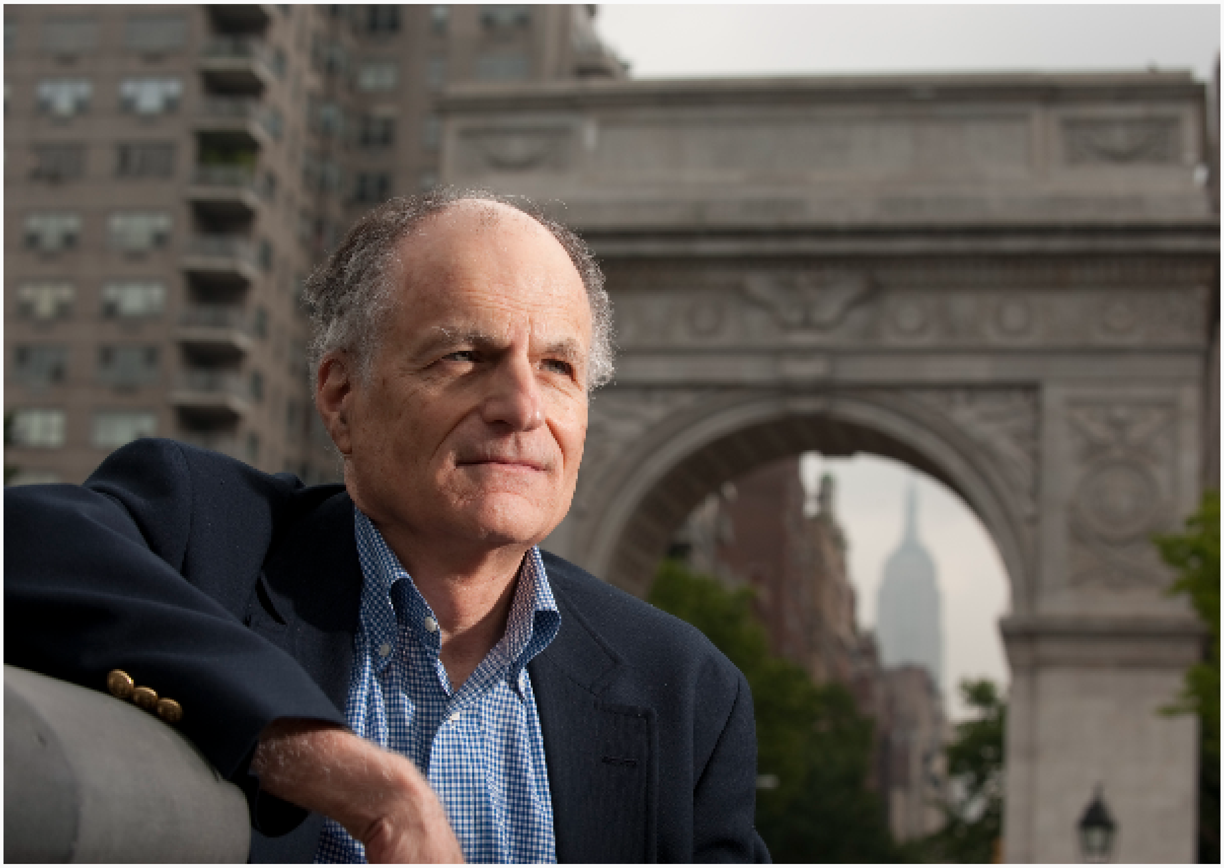
袁亚萍勉励大家要做到四个“勤”。一是“眼勤”，时刻关心孩子们的情绪问题及心理健康状况，通过每天的陪伴，了解孩子们的天赋、秉性、爱好，做到因材施教；二是“口勤”，无论是与孩子之间的交流，还是与师傅、家长之间的交流，一旦发现问题就要及时沟通解决；三是“手勤”，青年教师应带头完成班务工作，及时总结、反馈、提炼自己的教育教学经验；四是“脑勤”，对于课堂生成的问题，青年教师应多总结、多反思，创造性实施教育教学，逐步形成自己的教学思路、教学特色和教学风格。

愿师徒携手并进，用心、用情浇灌新小的每一朵蔷薇花，让蔷薇苑里的花儿朵朵绽放，同时也期待听到青年教师们带来的更多好消息。

附件3

诺贝尔经济学奖得主加盟上海纽约大学数学研究中心

来源：新民晚报 作者：郜阳 2021-09-28 09:17:38



图说：2011年诺贝尔经济学奖得主托马斯·萨金特 上海纽约大学供图

新民晚报讯（记者 郜阳）记者从上海纽约大学获悉，2011年诺贝尔经济学奖得主、纽约大学经济学与商学教授托马斯·萨金特 （Thomas J. Sargent）正式成为上海纽约大学联聘教授，以及华东师范大学 -纽约大学数学联合研究中心教授。萨金特是当今宏观经济学、货币经济学与时间序列计量经济学等领域的泰斗级人物，他的加盟将给上纽大在金融数学领域的研究带来强大助力。

“对于萨金特教授加入我们的中心，我们感到非常荣幸。”研究中心联合主任、上纽大数学学科负责人、数学教授皮埃尔·塔雷斯表示，“萨金特教授将与我们一起投入数学与经济学的交叉学科研究当中，尤其是在金融数学领域。他的研究重点是运用平均场博弈论的特定衍生工具，对在大群体中相互作用的小个体的战略决策进行数学分析。”

萨金特说，他非常期待与数学中心的合作，向上海纽约大学各位数学同事，特别是皮埃尔·塔雷斯教授以及数学与数据科学助理教授马修·洛利埃两位学者学习。“活到老学到老！作为一名‘终身学习者’，我要向这两位相关领域的专家学习。”萨金特表示，“他们精通运用随机过程、机器学习、逼近理论与偏微分方程等工具来解决宏观经济学中尚未解决的问题，这也正是我的研究兴趣所在。他们都极富耐心，给予了我许多帮助。”

记者了解到，2002年，纽约大学文理学院经济学系和斯特恩商学院，共同任命托马斯·萨金特为学校首位威廉·柏克利经济和商业讲席教授。加盟纽约大学之前，他曾在斯坦福大学、芝加哥大学、明尼苏达大学等世界名校任教；1997年获得内默斯经济学奖，1983年当选美国国家科学院以及美国人文与科学院双院士。2011年，萨金特与其合作者、普林斯顿大学的克里斯托弗·西姆斯教授凭借“对宏观经济成因和效果所投入的实证研究”获得诺贝尔经济学奖。萨金特主要研究经济政策的变动对经济的影响，并开发了一套新的方法研究政策与经济之间的关系。

一直以来，萨金特主张运用最先进的数学方法研究经济学，并教导他的学生，“数学能帮助你在大学四年里理解经济学课程的真谛，为你打开经济学的大门，助力你未来的职业发展。作为一名本科生，如果你渴望全面深入地学习经济学，那你最好再选一两门数学或统计学课程，这可能比你花时间精力写出一篇优秀毕业论文更有价值。”

受疫情影响，萨金特暂时不能来上海，但他表示之前曾多次到上海参加学术会议，对中国和上海都非常熟悉了。“我结识了许多在中国的高校里任教的杰出经济学家，其中有不少就在上海。”萨金特说，“中国经济持续高速增长是一个前所未有的奇迹，这也成为了我所在领域的学者们的重点研究方向。我十分期待能够深入观察和了解中国及周边亚洲国家的经济发展。”

虽然还没有在上纽大教过课，萨金特已经在纽约大学及纽大的阿布扎比校园结识了许多在那里海外学习的上纽大学生，并为他们授课。“他们都受过严格的学术训练，极为聪慧，具备出色的学术素养。”

托马斯·萨金特也是自上纽大成立以来加盟学校的第三位诺奖得主。另外两位分别是上海纽约大学波动研究所联席所长、纽约大学斯特恩商学院金融学教授罗伯特·恩格尔，以及参与创建并持续推动学校课程体系与科研进程的纽约大学斯特恩商学院经济学教授保罗·罗默。

华东师范大学-纽约大学数学联合研究中心（上海纽约大学）成立于2013年，致力于打造一个特点鲜明的研究机构，推动现代数学及其应用的发展，并与纽约大学库朗数学科学研究所和华东师范大学数学系保持着紧密的合作关系。联合研究中心承袭了库朗研究所自1935年创立以来的标志性数学研究模式——汇聚了一群优秀、敬业且才华横溢的数学家，研究具有重要科学、技术、社会和经济学价值的问题，探索通过数学与计量工具来解决这些领域问题的方法。中心的主要研究方向包括概率论、非线性偏微分方程、流体力学、生物学、材料科学、计算神经科学和金融数学等领域的理论及应用研究。

编辑：钱文婷

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推荐阅读

上海普陀区19日、22日将开展2次全员核酸筛查 (//wap.xinmin.cn/content/32249881.html)

2022-10-18

(//wap.xinmin.cn/content/32

党的二十大举行第二场记者招待会介绍全面从严治党成效 (//wap.xinmin.cn/content/32250164.html)

2022-10-18

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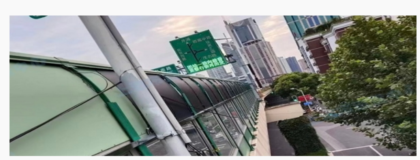
绿色低碳新做法！上海高架路声屏障试点安装太阳能光伏板 (//wap.xinmin.cn/content/32250162.html)

2022-10-18



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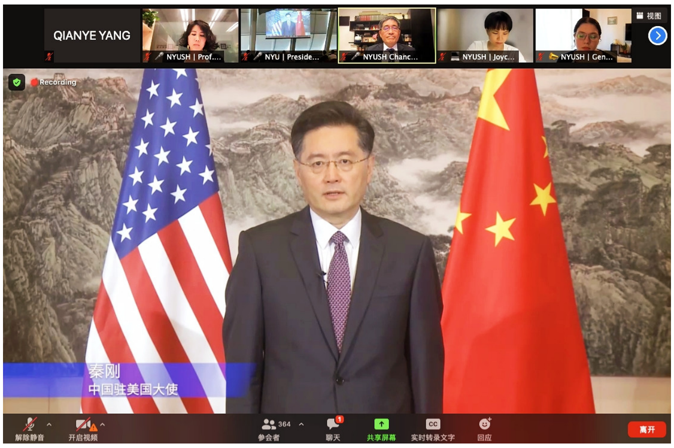
附件4

秦刚大使在上海纽约大学2021级新生欢迎仪式上发表视频致辞

2021/09/14 23:00

9月14日，中国驻美国大使秦刚以视频方式在上海纽约大学2021级新生欢迎仪式上致辞。



秦大使首先祝贺同学们即将在上海纽约大学开启新的人生旅程。他指出，当今世界，国与国日益成为相互依存、休戚与共的命运共同体。人文交流是国与国关系深化发
一如既往地支持中美教育交流，鼓励双向留学，深化高校合作。上海纽约大学是中美合作举办的第一所研究型大学，孔子说，“三人行，必有我师”，希望同学们能够用欣赏、
的师生们沟通交流，彼此成为良师益友，成为促进中外人文交流的使者。祝愿同学们学业有成。

上海纽约大学成立于2012年，是经中国教育部批准，由华东师范大学和纽约大学联合建立的合作办学机构，也是纽约大学全球体系中具有学位授予资格的三大校园之一

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附件5：荣誉数学拔尖创新人才培养成果典型案例 -- 2017届毕业生

夏家铭在博士在读期间与导师丁剑合著发表数学领域

顶级学术期刊 Inventiones Mathematicae 上的论文

夏家铭是首届（2017届）荣誉数学专业毕业生。学习期间修读独立学习课程，在 Charles Newman 教授一对一的指导下，对概率论方面的学习产生了浓厚的兴趣。毕业后获全额奖学金前往美国宾夕法尼亚大学攻读应用数学博士学位，师从知名概率论数学家丁剑教授。博士学习期间刻苦钻研，在概率论领域的研究中展现出卓越实力，她与导师合著的论文“Exponential decay of correlations in the two-dimensional random field Ising model”共同一作的身份在知名数学期刊 Inventiones Mathematicae 上。

Invent. math. (2021) 224:999–1045
https://doi.org/10.1007/s00222-020-01024-y

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Exponential decay of correlations in the two-dimensional random field Ising model

Jian Ding[1] · **Jiaming Xia**[1]

Received: 17 March 2019 / Accepted: 25 November 2020 / Published online: 1 January 2021

Abstract We study the random field Ising model on $\mathbb{Z}^2$ where the external field is given by i.i.d. Gaussian variables with mean zero and positive variance. We show that the effect of boundary conditions on the magnetization in a finite box decays exponentially in the distance to the boundary.

1 Introduction

For $v \in \mathbb{Z}^2$, let h_v be i.i.d. Gaussian variables with mean zero and variance $\varepsilon^2 > 0$. We consider the random field Ising model (RFIM) with external field $\{h_v : v \in \mathbb{Z}^2\}$ at temperature $T = 1/\beta \in [0, \infty)$. For $N \geqslant 1$, let $\Lambda_N = \{v \in \mathbb{Z}^2 : |v|_\infty \leqslant N\}$ be a box in $\mathbb{Z}^2$ centered at the origin o and of side length $2N$. For any set $A \subset \mathbb{Z}^2$, define $\partial A = \{v \in \mathbb{Z}^2 \setminus A : u \sim v \text{ for some } u \in A\}$ (where $u \sim v$ if $|u - v|_1 = 1$). The RFIM Hamiltonian $H^{\Lambda_N, \pm}$ on the configuration space $\{-1, 1\}^{\Lambda_N}$ with plus (respectively, minus) boundary condition and external field $\{h_v : v \in \Lambda_N\}$ is defined to be

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$$H^{\Lambda_N,\pm}(\sigma) = -\Big(\sum_{u\sim v, u,v\in\Lambda_N} \sigma_u\sigma_v \pm \sum_{u\sim v, u\in\Lambda_N, v\in\partial\Lambda_N} \sigma_u + \sum_{u\in\Lambda_N} \sigma_u h_u \Big) \quad (1)$$

for $\sigma \in \{-1, 1\}^{\Lambda_N}$. (In the preceding summation, each unordered pair $u \sim v$ only appears once.) Quenched on the external field $\{h_v\}$, the Ising measure with plus boundary condition (respectively minus boundary condition) is defined such that for all $\sigma \in \{-1, 1\}^{\Lambda_N}$ (throughout the paper the temperature is fixed, and thus we suppress the dependence on β in all notations)

$$\mu^{\Lambda_N,\pm}(\sigma) = \frac{e^{-\beta H^{\Lambda_N,\pm}(\sigma)}}{Z^{\Lambda_N,\pm}}, \text{ where } \quad Z^{\Lambda_N,\pm} = \sum_{\sigma'\in\{-1,1\}^{\Lambda_N}} e^{-\beta H^{\Lambda_N,\pm}(\sigma')}. \quad (2)$$

Note that $\mu^{\Lambda_N,\pm}$ is a random measure which itself depends on $\{h_v\}$. To be clear of the two different sources of randomness, we use $\mathbb{P}$ and $\mathbb{E}$ to refer to the probability measure with respect to the external field $\{h_v\}$; and we use $\mu^{\Lambda_N,\pm}$ for the Ising measures and use $\langle\cdot\rangle_{\mu^{\Lambda_N,\pm}}$ to denote the expectations with respect to the Ising measures.

Theorem 1.1 *For any $\varepsilon > 0$, $T \in [0, \infty)$, there exists $c = c(\varepsilon, T) > 0$ such that*

$$\mathbb{E}(\langle\sigma_o\rangle_{\mu^{\Lambda_N,+}} - \langle\sigma_o\rangle_{\mu^{\Lambda_N,-}}) \leqslant c^{-1}e^{-cN} \quad \textit{for all } N \geqslant 1.$$

This result lies under the umbrella of the general Imry–Ma [17] phenomenon, which states that in two-dimensional systems any first order transition is rounded off upon the introduction of arbitrarily weak static, or quenched, disorder in the parameter conjugate to the corresponding extensive quantity. In the particular case of the RFIM, it was shown in [4,5] that the effect of the boundary conditions on magnetization at distance N decays to 0 as $N \to \infty$ for all non-negative temperatures and arbitrarily weak quenched disorder (this also implies the uniqueness of the Gibbs state). The decay rate was then improved to $1/\sqrt{\log\log N}$ in [10] and to $1/N^\gamma$ (for some $\gamma > 0$) in [3]. In the presence of strong disorder it has been shown that there is an exponential decay [6,9,14] (see also [3, Appendix A]). The main remaining challenge is to decide whether the decay rate is exponential when the disorder is weak. In fact, there have been debates even among physicists as to whether there exists a regime where the decay rate is polynomial, and weak supporting arguments have been made in both directions [7,12,15]—in particular in [12] an argument was made for polynomial decay *at zero temperature* for a certain choice of disorder. Theorem 1.1 provides a complete answer to this question when the random field consists of i.i.d. Gaussian variables.

The two-dimensional behavior of the RFIM is drastically different from that for dimensions three and higher: it was shown in [16] that at zero temperature the effect on the local quenched magnetization of the boundary conditions at distance N does not vanish in N in the presence of weak disorder, and later an analogous result was proved in [8] at low temperatures. A heuristic explanation behind the different behaviors is as follows: in d dimensions the fluctuation of the random field in a box of side length N is of order $N^{d/2}$, whereas boundary condition effect is of order N^{d-1} (thus, in two dimensions the fluctuation of the random field in a box is of the same order as the size of the boundary, while in three dimensions and above the fluctuation of the random field is substantially smaller than the size of the boundary).

Our proof method is different from all of [3,5,10] (and different from [6, 9,14]), except that in the heuristic level our proof seems to be related to the Mandelbrot percolation analogy presented in [3, Appendix B]. The works [4,5] treated a wide class of distributions for disorder, while [3,10] and this paper work with Gaussian disorder. The main features of Gaussian distributions used in this paper are the simple formula for the change of measure [see (14)] and linear decompositions for Gaussian process [see (23)]. In addition, we remark that the analysis in [3,5] extends to the case with finite-range interactions. While we expect our framework to be useful in analyzing the finite-range case, the lack of planar duality seems to present some non-trivial obstacle (see Remark 2.3).

The rest of the paper consists of two sections. In Sect. 2, we prove Theorem 1.1 in the special case of $T = 0$. In our opinion, this is a significant simplification of the general case but still captures the core challenge of the problem. We hope that some of the key ideas (e.g., the crucial application of [1]) can be more transparent by first presenting the proof in this simplified case. In Sect. 3, we then present the proof for the case of $T > 0$. While the proof naturally shares the key insights with the case for $T = 0$, it seems to us that there are significant additional obstacles. As a result, the proof is not presented as an extension of the zero-temperature case. Instead, we present an almost self-contained proof, but omit details at times when they are merely adaption of arguments in Sect. 2.

Our (shared) notations in Sects. 2 and 3 are consistent with each other, and a few notations in Sect. 3 are natural extensions of those in Sect. 2. However, for clarity of exposition, we will recall or re-explain all notations in Sect. 3.

Concurrent work During the submission of this paper, a paper [2] which proved the same result was completed. The proof of [2] was inspired by the proof at zero temperature in this paper (for the crucial application of [1]). Both proofs share the basic intuition of "using the fluctuation of the sum of the random field in a box to fight the influence of the boundary condition" (which went back to [4,5]) and both apply [1] to disagreement percolation in a crucial manner. However, the two approaches seem to be rather different in at least the following two important aspects: (1) This paper employs first moment analysis via various perturbations of the random field, and the paper [2] (similar to [3]) relies on concentration/anti-concentration type of analysis (which in particular uses second-moment computations); (2) At positive temperatures, this paper employs a certain monotone coupling (adaptive admissible coupling as in Definition 3.9) between Ising measures with different boundary conditions, and the paper [2] considers a continuous extension of the Ising model into the metric graph which allows to study spin correlations via disagreement percolation for two *independent* samples (inspired by [18, 19]).

2 Exponential decay at zero temperature

At zero temperature, $\mu^{\Lambda_N,+}$ (and respectively $\mu^{\Lambda_N,-}$) is supported on the minimizer of (1), which is known as the *ground state* and is unique with probability 1. We denote by $\sigma^{\Lambda_N,+}$ the ground state with respect to the plus-boundary condition and by $\sigma^{\Lambda_N,-}$ the ground state with respect to the minus-boundary condition. Therefore, for $T = 0$ we have the simplification that the only randomness is from the $\mathbb{P}$-measure. Thus, Theorem 1.1 for $T = 0$ can then be simplified as follows.

Theorem 2.1 *For any $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that $\mathbb{P}(\sigma_o^{\Lambda_N,+} \neq \sigma_o^{\Lambda_N,-}) \leqslant c^{-1}e^{-cN}$ for all $N \geqslant 1$.*

2.1 Outline of the proof

We first reformulate Theorem 2.1. For $v \in \Lambda_N$, we define

$$\xi_v^{\Lambda_N} = \begin{cases} + & \text{if } \sigma_v^{\Lambda_N,+} = \sigma_v^{\Lambda_N,-} = 1, \\ - & \text{if } \sigma_v^{\Lambda_N,+} = \sigma_v^{\Lambda_N,-} = -1, \\ \mathsf{0} & \text{if } \sigma_v^{\Lambda_N,+} = 1 \text{ and } \sigma_v^{\Lambda_N,-} = -1. \end{cases} \tag{3}$$

By monotonicity (c.f. [3, Section 2.2]), the case of $\sigma_v^{\Lambda_N,+} = -1$ and $\sigma_v^{\Lambda_N,-} = 1$ cannot occur, so $\xi_v^{\Lambda_N}$ is well-defined for all $v \in \Lambda_N$. Theorem 2.1 can be restated as

$$m_N \leqslant c^{-1}e^{-cN} \quad \text{for} \quad c = c(\varepsilon) > 0, \quad \text{where} \quad m_N \triangleq \mathbb{P}(\xi_o^{\Lambda_N} = \mathsf{0}). \tag{4}$$

For any $A \subset \mathbb{Z}^2$, we can analogously define ξ^A by replacing Λ_N with A in (1) and (3). Let $\mathcal{C}^A = \{v \in A : \xi_v^A = \mathsf{0}\}$ (that is, $\mathcal{C}^A$ is the collection of *disagreements*). Monotonicity (see [3, (2.7)]) implies that

$$\mathcal{C}^B \cap B' \subset \mathcal{C}^{B'} \text{ provided that } B' \subset B. \tag{5}$$

In particular, this implies that m_N is decreasing in N, so we need only consider $N = 2^n$ for $n \geqslant 1$. Clearly, for any $v \in \mathcal{C}^A$, there exists a path in $\mathcal{C}^A$ joining v and ∂A. This suggests consideration of percolation properties of $\mathcal{C}^A$. Indeed, a key step in our proof for (4) is the following proposition on the lower bound on the length exponent for geodesics (i.e., shortest paths) in $\mathcal{C}^{\Lambda_N}$. For any $A \subset \mathbb{Z}^2$, we denote by $d_A(\cdot, \cdot)$ the intrinsic distance on A, i.e., the graph distance on the induced subgraph on A. Let $d_A(A_1, A_2) = \min_{x \in A_1 \cap A, y \in A_2 \cap A} d_A(x, y)$ (with the convention that $\min \emptyset = \infty$).

Proposition 2.2 *There exist $\alpha = \alpha(\varepsilon) > 1$, $\kappa = \kappa(\varepsilon) > 0$ such that for all $N \geqslant 1$*

$$\mathbb{P}(d_{\mathcal{C}^{\Lambda_N}}(\partial \Lambda_{N/4}, \partial \Lambda_{N/2}) \leqslant N^\alpha) \leqslant \kappa^{-1} e^{-N^\kappa}. \tag{6}$$

Remark 2.3 The "only" place where our proof breaks in extending to the finite range case is to verify Proposition 2.2 (and its analogue at positive temperatures, Proposition 3.1). The exact points where the extension of the proof encounters issues depend somewhat on exact formulations for sub-lemmas. For instance, at zero temperature one can try to prove a version of Lemma 2.8 sticking to nearest neighbor crossings, then for lack of planar duality there are issues both in the Proof of Lemma 2.8 (more specifically in Case 1) and in the Proof of (8) which applies Lemma 2.8. Of course one can also try to prove a stronger version of Lemma 2.8 (which suffices to prove (8)), but this may be hard.

The Proof of Proposition 2.2 will rely on [1], which takes the next lemma as input. For any rectangle $A \subset \mathbb{R}^2$ (whose sides are not necessarily parallel to the axes), let ℓ_A be the length of the longer side and let A^{Large} be (the lattice points of) the square box concentric with A, of side length $32\ell_A$ and with sides parallel to axes. In addition, define the aspect ratio of A to be the ratio between the lengths of the longer and shorter sides. For a (random) set $\mathcal{C} \subset \mathbb{Z}^2$, we use $\mathrm{Cross}(A, \mathcal{C})$ to denote the event that there exists a path $v_0, \ldots, v_k \in A \cap \mathcal{C}$ connecting the two shorter sides of A (that is, v_0, v_k are of ℓ_∞-distances less than 1 respectively from the two shorter sides of A).

Lemma 2.4 *Write $a = 100$. There exists $\ell_0 = \ell_0(\varepsilon)$ and $\delta = \delta(\varepsilon) > 0$ such that the following holds for any $N \geqslant 1$. For any $k \geqslant 1$ and any rectangles $A_1, \ldots, A_k \subseteq \{v \in \mathbb{R}^2 : |v|_\infty \leqslant N/2\}$ with aspect ratios at least a such that (a) $\ell_0 \leqslant \ell_{A_i} \leqslant N/32$ for all $1 \leqslant i \leqslant k$ and (b) $A_1^{\text{Large}}, \ldots, A_k^{\text{Large}}$ are* *disjoint, we have*

$$\mathbb{P}(\cap_{i=1}^{k}\mathrm{Cross}(A_i, \mathcal{C}^{\Lambda_N})) \leqslant (1-\delta)^k.$$

(Actually, the authors of [1] treated random curves in $\mathbb{R}^2$. However, the main capacity analysis can be copied in the discrete case, and the connection between the capacity and the box-counting dimension is straightforward (c.f. [11, Lemma 2.3]).) Armed with Lemma 2.4, we can apply [1, Theorem 1.3] to deduce that for some $\alpha = \alpha(\varepsilon) > 1$,

$$\mathbb{P}(d_{\mathcal{C}^{\Lambda_N}}(\partial\Lambda_{N/4}, \partial\Lambda_{N/2}) \leqslant N^{\alpha}) \to 0 \quad \text{as} \quad N \to \infty. \tag{7}$$

By a standard percolation argument (Lemma 2.10) which we will explain later, we can enhance the probability decay in (7) and prove (6).

By (5), the random set $\mathcal{C}^{\Lambda_N} \cap A$ is stochastically dominated by $\mathcal{C}^{A^{\mathrm{Large}}} \cap A$ as long as $A^{\mathrm{Large}} \subset \Lambda_N$. Moreover, it is obvious that $\mathcal{C}^{A_i^{\mathrm{Large}}}$ for $1 \leqslant i \leqslant k$ are mutually independent, as long as the sets A_i^{Large} for $1 \leqslant i \leqslant k$ are disjoint. Therefore, in order to prove Lemma 2.4, it suffices to show that for any rectangle A with aspect ratio at least $a = 100$ we have

$$\mathbb{P}(\mathrm{Cross}(A, \mathcal{C}^{A^{\mathrm{Large}}})) \leqslant 1-\delta \quad \text{where} \quad \delta = \delta(\varepsilon) > 0. \tag{8}$$

Both the Proof of (8) and the application of (6) rely on a perturbative analysis, which is another key feature of our proof. Roughly speaking, the logic is as follows:

- We first consider the perturbation by increasing the field by an amount of order $1/N$, and use this to show that the probability for a $\mathsf{0}$-valued contour surrounding an annulus is strictly bounded away from 1.
- Based on this property, we prove (8), which then implies (6).
- Given (6), we then show that increasing the field by an amount of order $1/N^{\alpha}$ (recall that $\alpha > 1$ is from Proposition 2.2 and thus the perturbation here is $1/N^{\alpha} \ll 1/N$) will most likely change the $\mathsf{0}$'s to $+$'s. Based on this, we prove polynomial decay for m_N with large power, which can then be enhanced to exponential decay.

For compactness of exposition, the actual implementation will differ slightly from the above plan:

- We first prove a general perturbation result (Lemma 2.5) in Sect. 2.2, where the size of perturbation is related to the intrinsic distance on $\mathcal{C}^{\Lambda_N}$.
- In Sect. 2.3, we apply Lemma 2.5 by bounding $d_{\mathcal{C}^{\Lambda_N}}$ from below by the ℓ_1-distance and correspondingly setting the perturbation amount to $1/N$, thereby proving Lemma 2.8. As a consequence, we verify (8).

- In Sect. 2.4, we apply Lemma 2.5 again by applying a lower bound on $d_{\mathcal{C}^{\Lambda_N}}$ from Proposition 2.2. This allows us to derive Lemma 2.11. As a consequence, we prove in Lemma 2.14 polynomial decay for m_N with large power, which is then enhanced to exponential decay by a standard argument.

2.2 A perturbative analysis

We first introduce some notation. For $A \subseteq \mathbb{Z}^2$, we set $h_A = \sum_{v \in A} h_v$. For $A, B \subset \mathbb{Z}^2$, we denote by $E(A, B) = \{\langle u, v\rangle : u \sim v, u \in A, v \in B\}$. Note that we treat $\langle u, v\rangle$ as an ordered edge. For simplicity, we will only consider $N = 2^n$ for $n \geqslant 10$. Let $\mathcal{A}_N = \Lambda_N \setminus \Lambda_{N/2}$ be an annulus. Define $\{\tilde{h}_v^{(N)} : v \in \Lambda_N\}$ to be a perturbation of the original field parameterized by $\Delta > 0$, as follows:

$$\tilde{h}_v^{(N)} = h_v + \Delta \quad \text{for } v \in \Lambda_N. \tag{9}$$

We will use $\tilde{H}^{\Lambda_N,\pm}(\sigma)$, $\tilde{\sigma}^{\Lambda_N,\pm}$, $\tilde{\xi}^{\Lambda_N}$, $\tilde{\mathcal{C}}^{\Lambda_N}$ to denote the corresponding tilde versions of $H^{\Lambda_N,\pm}(\sigma)$, $\sigma^{\Lambda_N,\pm}$, ξ^{Λ_N}, $\mathcal{C}^{\Lambda_N}$ respectively, i.e., defined analogously but with respect to the field $\{\tilde{h}_v^{(N)}\}$. In addition, define $\mathcal{C}_*^{\Lambda_N} = \tilde{\mathcal{C}}^{\Lambda_N} \cap \mathcal{C}^{\Lambda_N}$ (so $\mathcal{C}_*^{\Lambda_N}$ is the intersection of disagreements with respect to the original and the perturbed field; in informal discussions we will refer to vertices in $\mathcal{C}_*^{\Lambda_N}$ as disagreements too).

Lemma 2.5 *Consider $K, \Delta > 0$. Define $\{\tilde{h}_v^{(N)} : v \in \Lambda_N\}$ as in (9). The following two conditions cannot hold simultaneously:*

(a) $d_{\mathcal{C}_*^{\Lambda_N}}(\partial\Lambda_{N/4}, \partial\Lambda_{N/2}) \geqslant K$;

(b) $|\mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/4}| \cdot \Delta > \frac{8}{K}|\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}|$.

Proof Suppose otherwise both (a) and (b) hold. Let $B_k = \{v \in \mathcal{A}_{N/2} : d_{\mathcal{C}_*^{\Lambda_N}}(\partial\Lambda_{N/4}, v) = k\}$ for $k = 1, \ldots, K$. Note that $B_k \subset \mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}$ for all $1 \leqslant k \leqslant K$ by (a). It is obvious that the B_k's are disjoint from each other, and thus there exists a minimal value k_* such that

$$|B_{k_*}| \leqslant K^{-1}|\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}|. \tag{10}$$

Let

$$S = (\mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/4}) \cup \cup_{k=1}^{k^*-1} B_k,$$

and for $\tau \in \{-, 0, +\}$, define

$$g(S,\tau) = \{\langle u,v\rangle \in E(S,S^c) : \xi_v^{\Lambda_N} = \tau\} \quad \text{and}$$
$$\tilde{g}(S,\tau) = \{\langle u,v\rangle \in E(S,S^c) : \tilde{\xi}_v^{\Lambda_N} = \tau\}. \tag{11}$$

Note that for any $v \in \Lambda_N$ with $\xi_v^{\Lambda_N} = 0$ we have $\sigma_v^{\Lambda_N,+} = 1$. Since $\xi_v^{\Lambda_N} = 0$ for $v \in S$ (which implies that $\sigma_v^{\Lambda_N,+} = 1$ for $v \in S$),

$$h_S + |g(S,+)| - |g(S,-)| + |g(S,0)| \geqslant 0, \tag{12}$$

because if (12) does not hold, then $H^{\Lambda_N,+}(\sigma') < H^{\Lambda_N,+}(\sigma^{\Lambda_N,+})$ where σ' is obtained from $\sigma^{\Lambda_N,+}$ by flipping its value on S, thus contradicting the minimality of $H^{\Lambda_N,+}(\sigma^{\Lambda_N,+})$. In addition, by monotonicity (with respect to the external field), we have $g(S,0) \subset \tilde{g}(S,0) \cup \tilde{g}(S,+)$, $g(S,+) \subset \tilde{g}(S,+)$, and thus

$$|\tilde{g}(S,+)| - |g(S,+)| \geqslant |g(S,0) \setminus \tilde{g}(S,0)|.$$

Similarly, we have $\tilde{g}(S,-) \subset g(S,-)$ and $\tilde{g}(S,0) \subset g(S,-) \cup g(S,0)$, and thus

$$|g(S,-)| - |\tilde{g}(S,-)| \geqslant |\tilde{g}(S,0) \setminus g(S,0)|.$$

By our definition of B_k's, we see that $\tilde{g}(S,0) \cap g(S,0) = E(S, B_{k_*})$. Therefore, (12) and the preceding two displays imply that

$$\begin{aligned}
&\tilde{h}_S^{(N)} + |\tilde{g}(S,+)| - |\tilde{g}(S,-)| - |\tilde{g}(S,0)| \\
&\quad \geqslant \tilde{h}_S^{(N)} + |g(S,+)| - |g(S,-)| + |g(S,0)| - 2|E(S,B_{k_*})| \\
&\quad \geqslant |S|\Delta - 8|B_{k_*}| > 0,
\end{aligned}$$

where the last inequality follows from (b) and (10). The preceding inequality implies $\tilde{H}^{\Lambda_N,-}(\sigma') < \tilde{H}^{\Lambda_N,-}(\tilde{\sigma}^{\Lambda_N,-})$ where σ' is obtained from $\tilde{\sigma}^{\Lambda_N,-}$ by flipping its value on S. This contradicts the minimality of $\tilde{H}^{\Lambda_N,-}(\tilde{\sigma}^{\Lambda_N,-})$, completing the proof of the lemma. □

Lemma 2.6 *For any $x_v \geqslant 0$ for $v \in \Lambda_N$, let $\check{h}_v^{(N)} = h_v + x_v$ for $v \in \Lambda_N$ (we will use $\check{H}^{\Lambda_N,\pm}(\sigma)$, $\check{\sigma}^{\Lambda_N,\pm}$, $\check{\xi}^{\Lambda_N}$, $\check{\mathcal{C}}^{\Lambda_N}$ to denote the corresponding$\check{}$ versions of $H^{\Lambda_N,\pm}(\sigma)$, $\sigma^{\Lambda_N,\pm}$, ξ^{Λ_N}, $\mathcal{C}^{\Lambda_N}$). Then with probability 1, for any $v \in \mathcal{C}^{\Lambda_N} \cap \check{\mathcal{C}}^{\Lambda_N}$ there is a path in $\mathcal{C}^{\Lambda_N} \cap \check{\mathcal{C}}^{\Lambda_N}$ joining v and $\partial\Lambda_N$.*

Proof The proof is similar to that of Lemma 2.5, and in a way it is the case of $K = \infty$ there.

Suppose that the claim is not true. Then take $v \in \mathcal{C}^{\Lambda_N} \cap \check{\mathcal{C}}^{\Lambda_N}$ (for which the claim fails), and let S be the connected component in $\mathcal{C}^{\Lambda_N} \cap \check{\mathcal{C}}^{\Lambda_N}$ that contains v (thus S is not neighboring $\partial\Lambda_N$). Define $g(S,\tau)$ as in (11) and define $\check{g}(S,\tau) = \{\langle u, v\rangle \in E(S,S^c) : \check{\xi}_v^{\Lambda_N} = \tau\}$. Similar to (12), we have that

$$h_S + |g(S,+)| - |g(S,-)| + |g(S,\mathsf{0})| \geqslant 0.$$

In our case, $g(S,\mathsf{0}) \cup g(S,+) \subset \check{g}(S,+)$ and $\check{g}(S,\mathsf{0}) \cup \check{g}(S,-) \subset g(S,-)$. Therefore,

$$\begin{aligned}&\check{h}_S^{(N)} + |\check{g}(S,+)| - |\check{g}(S,-)| - |\check{g}(S,\mathsf{0})|\\ &\quad\geqslant h_S + |g(S,+)| - |g(S,-)| + |g(S,\mathsf{0})| \geqslant 0.\end{aligned}$$

The preceding inequality implies that $\check{H}^{\Lambda_N,-}(\sigma') \leqslant \check{H}^{\Lambda_N,-}(\check{\sigma}^{\Lambda_N,-})$ where σ' is obtained from $\check{\sigma}^{\Lambda_N,-}$ by flipping its value on S. This happens with probability 0 since the ground state is unique with probability 1. □

2.3 Proof of Proposition 2.2

In this section, we will set $K = K(N) = N/4$, and $\Delta = \Delta(N) = \gamma/N$ for an absolute constant $\gamma > 0$ to be selected, and we consider $\tilde{h}^{(N)}$ as in (9). In this case Condition (a) in Lemma 2.5 holds trivially. For convenience, we use $\mathbb{P}_N$ to denote the probability measure with respect to the field $\{h_v : v \in \Lambda_N\}$ and use $\tilde{\mathbb{P}}_N$ to denote the probability measure with respect to $\{\tilde{h}_v^{(N)} : v \in \Lambda_N\}$.

Lemma 2.7 *Recall that ε is the variance parameter for the field $\{h_v\}$. For any $p > 0$, there exists $c = c(\varepsilon, p, \gamma) > 0$ such that for any event E_N with $\tilde{\mathbb{P}}_N(E_N) \geqslant p$, we have that*

$$\mathbb{P}_N(E_N) \geqslant c.$$

Proof There exists a constant $C > 0$ such that $\tilde{\mathbb{P}}_N(|\tilde{h}_{\Lambda_N}^{(N)} - \Delta|\Lambda_N|| \geqslant C\varepsilon N) \leqslant p/2$. Thus we have

$$\tilde{\mathbb{P}}_N(E_N; |\tilde{h}_{\Lambda_N}^{(N)} - \Delta|\Lambda_N|| \leqslant C\varepsilon N) \geqslant p/2. \tag{13}$$

Also, by a straightforward Gaussian computation, we see that

$$\frac{d\mathbb{P}_N}{d\tilde{\mathbb{P}}_N} = \exp\Big\{-\frac{\Delta(\tilde{h}_{\Lambda_N}^{(N)} - \Delta|\Lambda_N|)}{\varepsilon^2}\Big\}\exp\Big\{\frac{-\Delta^2|\Lambda_N|}{2\varepsilon^2}\Big\} \tag{14}$$

and thus there exists $\iota = \iota(\varepsilon) > 0$ such that

$$\frac{d\mathbb{P}_N}{d\tilde{\mathbb{P}}_N} \geqslant \iota \quad \text{provided that } |\tilde{h}^{(N)}_{\Lambda_N} - \Delta|\Lambda_N|| \leqslant C\varepsilon N.$$

Combined with (13), this completes the proof of the lemma. □

For any annulus $\mathcal{A}$, we denote by $\mathrm{Cross}_{\mathrm{hard}}(\mathcal{A}, \mathcal{C})$ the event that there is a contour in $\mathcal{C}$ which separates the inner and outer boundaries of $\mathcal{A}$, and by $\mathrm{Cross}_{\mathrm{easy}}(\mathcal{A}, \mathcal{C})$ the event that there is a path in $\mathcal{C}$ which connects the inner and outer boundaries of $\mathcal{A}$.

Lemma 2.8 *There exists* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\min\{\mathbb{P}(\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{C}^{\Lambda_N})), \mathbb{P}(\mathrm{Cross}_{\mathrm{easy}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{C}^{\Lambda_N}))\} \leqslant 1 - \delta \quad \textit{for all } N \geqslant 32.$$

Proof We first provide a brief discussion on the outline of the proof. We refer to the disagreements on $\Lambda_{N/32}$ with plus/minus boundary conditions posed on $\partial\Lambda_{N/8}$ as the "enhanced" disagreements (the word enhanced is chosen since the enhanced disagreements stochastically dominate the disagreements with boundary conditions on $\partial\Lambda_N$ by monotonicity of the Ising model). Note that the set of disagreements in $\mathcal{A}_{N/2}$ is stochastically dominated by the union of a constant number of copies of enhanced disagreements, which are independent of the enhanced disagreements in $\Lambda_{N/32}$. Therefore, with positive probability the number of enhanced disagreements in $\Lambda_{N/32}$ is larger than (up to a constant factor) the number of disagreements in $\mathcal{A}_{N/2}$ (see (16)). On this event, (modulo a caveat) by Lemma 2.5 at least one of the enhanced disagreements is not a disagreement when considering boundary conditions on $\partial\Lambda_N$ — this yields the desired statement as incorporated in **Case 1** below. In **Case 2**, we tighten the argument by addressing the caveat which is the scenario that the enhanced disagreement is empty (this is relatively simple).

We are now ready to carry out the formal proof. We can write $\mathcal{A}_{N/2} = \cup_{i=1}^{r} A_i$ where each A_i is a box of side length $N/16$ (so a copy of $\Lambda_{N/32}$) and $r \geqslant 16$ is a fixed integer (while it is conventional to choose A_i's as disjoint boxes, the disjointness is not used in the proof). For a box A, denoting by A^{Big} as the concentric box of A whose side length is $4\ell_A$. We have that (see Fig. 1)

$$A_i^{\mathrm{Big}} \cap \Lambda_{N/8} = \emptyset \quad \text{and } A_i^{\mathrm{Big}} \subset \Lambda_N \quad \text{for all } 1 \leqslant i \leqslant r. \tag{15}$$

For any $A \subset \Lambda_N$, let $\bar{\mathcal{C}}^A$ be defined as $\mathcal{C}^A$ but replacing $\{h_v : v \in A\}$ by $\{\tilde{h}_v^{(N)} : v \in A\}$ (note that $\bar{\mathcal{C}}^{\Lambda_{N/2}}$ is different from $\tilde{\mathcal{C}}^{\Lambda_{N/2}}$, which is defined with respect to $\tilde{h}^{(N/2)}$). Write $\mathcal{C}_\diamond^A = \mathcal{C}^A \cap \bar{\mathcal{C}}^A$. Write $X_i = |\mathcal{C}_\diamond^{A_i^{\mathrm{Big}}} \cap A_i|$ and

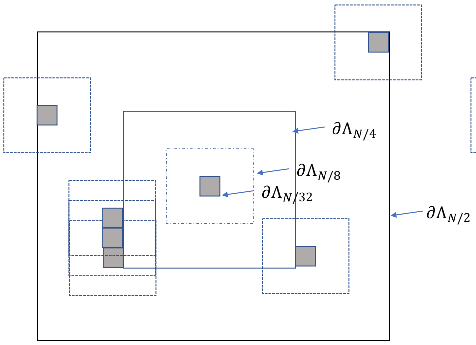

Fig. 1 Illustration for the geometric setup of the proof for Lemma 2.7. In the picture on the left we cover $\mathcal{A}_{N/2}$ by a collection of translated copies of $\Lambda_{N/32}$ (the grey boxes) — we only draw out a few copies for an illustration. Note that the (4-times) enlargements of translated copies (while overlapping among themselves) are all disjoint with $\Lambda_{N/8}$. The picture on the right illustrates the scenario in Case 1: for some $v \in \mathcal{C}_\diamond^{\Lambda_{N/8}} \setminus \mathcal{C}^{\Lambda_N}$, we draw its component with the same ξ^{Λ_N}-value and this component necessarily goes out of $\Lambda_{N/8}$

$X = |\mathcal{C}_\diamond^{\Lambda_{N/8}} \cap \Lambda_{N/32}|$. Clearly, X_i's and X are identically distributed and by (15) X_i's are independent of X (but X_i's are not mutually independent). Let $\theta = \inf\{x : \mathbb{P}(X \leqslant x) \geqslant 1 - 1/2r\}$. Thus,

$$\mathbb{P}(X \geqslant \max_{1\leqslant i\leqslant r} X_i, X \geqslant \theta) \geqslant \mathbb{P}(X \geqslant \theta)\mathbb{P}(\max_{1\leqslant i\leqslant r} X_i \leqslant \theta) \geqslant 1/4r. \quad (16)$$

The rest of the proof divides into two cases.

Case 1: $\theta > 0$. Let $\mathcal{E} = \{|\mathcal{C}_\diamond^{\Lambda_{N/8}} \cap \Lambda_{N/32}| \geqslant r^{-1}|\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}|\} \cap \{|\mathcal{C}_\diamond^{\Lambda_{N/8}} \cap \Lambda_{N/32}| > 0\}$. By (5) and (15), we have $|\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}| \leqslant \sum_{i=1}^r X_i$. Combined with (16), it gives that $\mathbb{P}(\mathcal{E}) \geqslant 1/4r$. Setting $\gamma = 100r$, we get that $|\mathcal{C}_\diamond^{\Lambda_{N/8}} \cap \Lambda_{N/32}| \cdot \Delta > 16K^{-1}|\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}|$ on $\mathcal{E}$. By Lemma 2.5, on $\mathcal{E}$ there is at least one vertex $v \in \mathcal{C}_\diamond^{\Lambda_{N/8}} \cap \Lambda_{N/32}$ but $v \notin \mathcal{C}_*^{\Lambda_N}$. So either $v \notin \mathcal{C}^{\Lambda_N}$ or $v \notin \tilde{\mathcal{C}}^{\Lambda_N}$ on $\mathcal{E}$. Assume that $v \notin \mathcal{C}^{\Lambda_N}$ and the other case can be treated similarly.

We will use the following property: for any connected set $\mathcal{A}$, $u \notin \mathcal{C}^{\mathcal{A}}$ if and only if there exists a connected set $A \subset \mathcal{A}$ with $u \in A$ such that $\xi_w^A = +$ for all $w \in A$ or $\xi_w^A = -$ for all $w \in A$. The "if" direction of the property follows from (5). For the "only if" direction, we assume without loss that $\xi_u^{\mathcal{A}} = +$ and let A be the connected component containing u where the $\xi^{\mathcal{A}}$-value is $+$. Note $\sigma_w^{\mathcal{A},-} = -1$ for all $w \in \partial A$ and $\sigma_w^{\mathcal{A},-} = 1$ for all $w \in A$. This implies that $\xi_w^A = +$ for all $w \in A$.

By the preceding property, there exists a connected set $A \subset \Lambda_N$ with $v \in A$ such that $\xi_w^A = +$ for all $w \in A$ or $\xi_w^A = -$ for all $w \in A$ (see Fig. 1 for an illustration). In addition, A cannot be contained in $\Lambda_{N/8}$ since otherwise it contradicts $v \in \mathcal{C}^{\Lambda_{N/8}}$. By planar duality, this implies that on $\mathcal{E}$, either

$\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{C}^{\Lambda_N})$ or $\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \tilde{\mathcal{C}}^{\Lambda_N})$ does not occur (the second case corresponds to the case when $v \notin \tilde{\mathcal{C}}^{\Lambda_N}$). Therefore,

$$\begin{aligned}&\mathbb{P}((\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{C}^{\Lambda_N}))^c)\\ &\quad+\mathbb{P}((\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \tilde{\mathcal{C}}^{\Lambda_N}))^c) \geqslant \mathbb{P}(\mathcal{E}) \geqslant 1/4r.\end{aligned}$$

Combined with Lemma 2.7, this completes the proof of the lemma.
Case 2: $\theta = 0$. Applying a simple union bound (by using 16 copies of $\Lambda_{N/32}$ to cover $\Lambda_{N/8}$, and a derivation similar to $|\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}| \leqslant \sum_{i=1}^r X_i$) we get that $\mathbb{P}(\mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/8} = \emptyset) \geqslant 1/2$. We assume without loss that $\mathbb{P}(\mathrm{Cross}_{\mathrm{easy}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{C}^{\Lambda_N})) \geqslant 3/4$ (otherwise there is nothing further to prove), and thus

$$\mathbb{P}(\mathrm{Cross}_{\mathrm{easy}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{C}^{\Lambda_N}) \text{ and } \mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/8} = \emptyset) \geqslant 1/4.$$

On the event $\mathrm{Cross}_{\mathrm{easy}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{C}^{\Lambda_N})$ and $\mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/8} = \emptyset$, the easy crossing (joining two boundaries of $\Lambda_{N/8} \setminus \Lambda_{N/32}$) in $\mathcal{C}^{\Lambda_N}$ becomes an easy crossing with $\tilde{\xi}^{\Lambda_N}$-values $+$. Thus, by planar duality, it prevents existence of a contour surrounding $\Lambda_{N/32}$ in $(\Lambda_{N/8} \setminus \Lambda_{N/32}) \cap \tilde{\mathcal{C}}^{\Lambda_N}$. Therefore,

$$\mathbb{P}((\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \tilde{\mathcal{C}}^{\Lambda_N}))^c) \geqslant 1/4.$$

Combined with Lemma 2.7, this completes the proof of the lemma. □

Proof of (8) Let $N = \min\{2^n : 2^{n+2} \geqslant \ell_A\}$. By our assumption on A, it is clear that we can position four copies A_1, A_2, A_3, A_4 of A by translation or rotation by 90 degrees so that (see the left of Fig. 2)

- $A_1, A_2, A_3, A_4 \subset \Lambda_{N/8} \setminus \Lambda_{N/32}$.
- The union of any crossings through A_1, A_2, A_3, A_4 in their longer directions surrounds $\Lambda_{N/32}$.
- $\Lambda_N \subset A_i^{\mathrm{Large}}$ for $1 \leqslant i \leqslant 4$.

Set $p = \mathbb{P}(\mathrm{Cross}(A, \mathcal{C}^{A^{\mathrm{Large}}}))$ (note that p depends on the dimension of A and also the orientation of A). By rotation symmetry and (5) we see that $\mathbb{P}(\mathrm{Cross}(A_i, \mathcal{C}^{\Lambda_N})) \geqslant \mathbb{P}(\mathrm{Cross}(A_i, \mathcal{C}^{A_i^{\mathrm{Large}}})) = p$. In what follows, we denote $\mathcal{A} = \Lambda_{N/8} \setminus \Lambda_{N/32}$. Then, by $\mathbb{P}(\mathrm{Cross}(A_i, \mathcal{C}^{\Lambda_N})) \geqslant p$ and a simple union bound, we get that

$$\mathbb{P}(\mathrm{Cross}_{\mathrm{hard}}(\mathcal{A}, \mathcal{C}^{\Lambda_N})) \geqslant \mathbb{P}(\cap_{i=1}^4 \mathrm{Cross}(A_i, \mathcal{C}^{\Lambda_N})) \geqslant 1 - 4(1-p). \quad (17)$$

Similarly, we can arrange two copies A_a, A_b of A obtained by translation and rotation by 90 degrees such that $\Lambda_N \subset A_a^{\mathrm{Large}}, A_b^{\mathrm{Large}}$ and that the union of any two crossings through A_a^{Large}, A_b^{Large} in the longer direction connects the two boundaries of $\mathcal{A}$ (see the right of Fig. 2). This implies that

$$\begin{aligned}&\mathbb{P}(\mathrm{Cross}_{\mathrm{easy}}(\mathcal{A}, \mathcal{C}^{\Lambda_N}))\\ &\quad \geqslant \mathbb{P}(\mathrm{Cross}(A_a, \mathcal{C}^{\Lambda_N}) \cap \mathrm{Cross}(A_b, \mathcal{C}^{\Lambda_N})) \geqslant 1 - 2(1-p). \qquad (18)\end{aligned}$$

Combined with (17) and Lemma 2.8, it yields that $p \leqslant 1 - \delta$ for some $\delta = \delta(\varepsilon) > 0$ as required. □

The following standard lemma will be applied several times below. Before presenting the lemma, we first provide a definition.

Definition 2.9 Divide Λ_N into disjoint boxes of side lengths $N' \leqslant N$ where $N' = 2^{n'}$ for some $n' \geqslant 1$, and denote by $\mathcal{B}(N, N')$ the collection of such boxes. Consider a percolation process on $\mathcal{B}(N, N')$, where each box $B \in \mathcal{B}(N, N')$ is regarded open or closed randomly. For $C, p > 0$, we say that the percolation process satisfies the (N, N', C, p)-condition if for each $B \in \mathcal{B}(N, N')$, there exists an event E_B such that

- On E_B^c, B is closed.
- $\mathbb{P}(E_B) \leqslant p$ for each B.
- If $\min_{x \in B_i, y \in B_j} |x - y|_\infty \geqslant CN'$ for all $1 \leqslant i < j \leqslant k$, then the events $E_{B_1}, \ldots, E_{B_k}$ are mutually independent.

Furthermore, we say two boxes B_1, B_2 are adjacent if $\min_{x_1 \in B_1, x_2 \in B_2} |x_1 - x_2|_\infty \leqslant 1$, and we say a collection of boxes is a lattice animal if these boxes form a connected graph.

Lemma 2.10 *For any $C > 0$, there exists $p > 0$ such that for all N and $N' \leqslant N$ and any percolation process on $\mathcal{B}(N, N', C, p)$ satisfying the (N, N', C, p)-condition, we have*

$$\begin{aligned}&\mathbb{P}(\textit{there exists a lattice animal of open boxes on } \mathcal{B}(N, N')\\ &\quad \textit{of size at least } k) \leqslant (\tfrac{N}{N'})^2 2^{-k}.\end{aligned}$$

Proof On the one hand, the number of lattice animals of size exactly k is bounded by $(\frac{N}{N'})^2 8^{2k}$ (the bound comes from first choosing a starting box, and then encoding the lattice animal by a surrounding contour on $\mathcal{B}(N, N')$ of length $2k$). On the other hand, for any k such boxes, we can extract a sub-collection of ck boxes (here $c > 0$ is a constant that depends only on C) such that the pairwise distances of boxes in this sub-collection are at least CN'; hence the probability that all these k boxes are open is at most p^{ck}. The proof of the lemma is then completed by a simple union bound, employing the (N, N', C, p)-condition. □

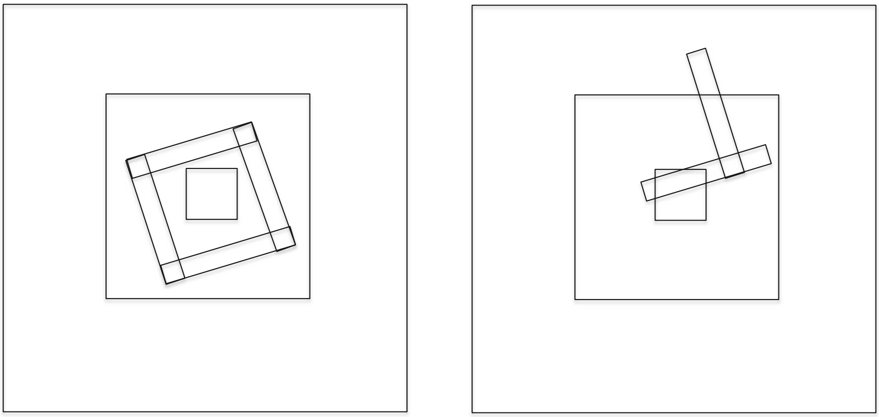

Fig. 2 On both left and right, the three concentric square boxes are Λ_N, $\Lambda_{N/8}$ and $\Lambda_{N/32}$ respectively. On the left, the four rectangles are A_1, A_2, A_3, A_4 and on the right the two rectangles are A_a, A_b

Proof of Proposition 2.2 Let $N' = N^{1-(\frac{\alpha-1}{10}\wedge\frac{1}{10})}$, where α is as in (7). For each $B \in \mathcal{B}(N, N')$, we say B is open if $d_{\mathcal{C}^{B^{\mathrm{Large}}}}(\partial B, \partial B^{\mathrm{large}}) \leqslant (N')^\alpha$, where B^{large} is the box concentric with B of doubled side length and B^{Large} (as we recall) is a concentric box of B with side length $32\ell_B$. By (7), we see that this percolation process satisfies the $(N, N', 64, p)$-condition where $p \to 0$ as $N \to \infty$. Now, in order that $d_{\mathcal{C}^{\Lambda_N}}(\partial\Lambda_{N/4}, \partial\Lambda_{N/2}) \leqslant (N')^\alpha$, there must exist an open lattice animal on $\mathcal{B}(N, N')$ of size at least $\frac{N}{16N'}$. Applying Lemma 2.10 completes the Proof of Proposition 2.2 (since $(\alpha(1-(\frac{\alpha-1}{10}\wedge\frac{1}{10})) > 1)$. $\square$

2.4 Proof of Theorem 2.1

In this subsection, we will show that the probability for $\{o \in \mathcal{C}^{\Lambda_N}\}$ has a polynomial decay with large power (Lemma 2.14), which then yields Theorem 2.1 by a standard application of Lemma 2.10. In order to prove Lemma 2.14, we first provide a bound on the probability for $\{o \in \mathcal{C}_*^{\Lambda_N}\}$ (Lemma 2.11), whose proof crucially relies on Proposition 2.2.

Let $\alpha > 1$ be as in Proposition 2.2 (note that we can assume without loss that $\alpha \leqslant 2$). Let $\sqrt{1/\alpha} < \alpha' < 1$ (and thus we have $\alpha(\alpha')^2 > 1$).

Lemma 2.11 *For $N^\diamond \geqslant 16$, set $\Delta = (N^\diamond)^{-\alpha(\alpha')^2}$ and let $\tilde{h}^{(N)}$ be defined as in (9) for $N \leqslant N^\diamond$. Write $m_N^\diamond = m_N^\diamond(N^\diamond) = \mathbb{P}(o \in \mathcal{C}_*^{\Lambda_N})$. Then there exists $C = C(\varepsilon) > 0$ such that $m_{N^\diamond}^\diamond \leqslant C(N^\diamond)^{-6}$.*

Remark 2.12 (1) In this lemma, regardless of the size of the box under consideration, the amount of perturbation Δ in our field $\tilde{h}^{(N)}$ only depends on

$N^\diamond$. This is crucial for (20) below. (2) Since $\alpha(\alpha')^2 > 1$, we have that $\Delta \ll 1/N^\diamond$ (this is crucial for getting a large power in the polynomial bound as in Lemma 2.14). (3) Since our perturbation $\Delta = (N^\diamond)^{-\alpha(\alpha')^2}$ applies to all $N \leqslant N^\diamond$, when N is very small in comparison of $N^\diamond$ the perturbation is possibly too mild and thus we may not have a good control on $\mathcal{C}_*^{\Lambda_N}$. However, this is not a problem because in the proof below we will only consider $N \geqslant (N^\diamond)^{\alpha'}$ (for which the perturbation is still significant).

Proof Write $K = (N^\diamond)^{\alpha\alpha'}$. We claim it suffices to show that there exists $N_0 = N_0(\varepsilon)$ such that for $N^\diamond \geqslant N_0$

$$m_{2N}^\diamond \leqslant K^{-\frac{1-\alpha'}{2}} m_{N/2}^\diamond \quad \text{for } (N^\diamond)^{\alpha'} \leqslant N \leqslant N^\diamond. \tag{19}$$

Indeed, since $K = (N^\diamond)^{\alpha\alpha'}$, we can deduce from (19) by recursion that $m_{N^\diamond}^\diamond \leqslant e^{-c(\log N^\diamond)^2}$ for some constant $c > 0$, which yields the claimed bound in the lemma (with room to spare).

We now turn to the Proof of (19). Suppose that (19) fails for some $(N^\diamond)^{\alpha'} \leqslant N \leqslant N^\diamond$. Since $\Lambda_N \subset v + \Lambda_{2N}$ for all $v \in \Lambda_{N/4}$ and $v + \Lambda_{N/2} \subset \Lambda_N$ for all $v \in \mathcal{A}_{N/2}$, by (5) we see

$$\mathbb{E}|\mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/4}| \geqslant \frac{N^2}{32} m_{2N}^\diamond \quad \text{and} \quad \mathbb{E}|\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}| \leqslant N^2 m_{N/2}^\diamond. \tag{20}$$

Together with the assumption that (19) fails, this yields that

$$\mathbb{E}|\mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/4}| > 32^{-1} K^{-\frac{1-\alpha'}{2}} \mathbb{E}|\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}|.$$

Since $|\mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/4}|$ and $|\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}|$ are integer-valued and are at most N^2, the preceding inequality implies that (recall that $\alpha' > 1/\sqrt{\alpha} \geqslant 1/\sqrt{2}$)

$$\mathbb{P}(|\mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/4}| > 64^{-1} K^{-\frac{1-\alpha'}{2}} |\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}|) \geqslant \frac{1}{32N^3}.$$

Now, set $N_0 = N_0(\varepsilon)$ sufficiently large so that

$$\frac{1}{10^6 N^3} > \kappa^{-1} e^{-N^\kappa} \text{and } 64^{-1} K^{-\frac{1-\alpha'}{2}} > \frac{8}{K\Delta} \text{ for all } N \geqslant (N_0)^{\alpha'}. \tag{21}$$

Therefore, by Proposition 2.2, there is a positive probability such that

$$|\mathcal{C}_*^{\Lambda_N} \cap \Lambda_{N/4}| > 64^{-1} K^{-\frac{1-\alpha'}{2}} |\mathcal{C}_*^{\Lambda_N} \cap \mathcal{A}_{N/2}| \quad \text{and}$$
$$d_{\mathcal{C}_*^{\Lambda_N}}(\partial\Lambda_{N/4}, \partial\Lambda_{N/2}) \geqslant K.$$

In particular, there exists at least one instance for the two events in the preceding display to occur simultaneously. This contradicts Lemma 2.5, thus completing the proof of the lemma. □

In the Proof of Lemma 2.14 below, it is important for us to have independence between different scales. To this end, it is useful to consider a perturbation which only occurs in an annulus. In order to make a difference in notation from the previous perturbation (which occurs in a whole box), for $\Delta(N) > 0$ we define (we emphasize the dependence of Δ on N in the notation here since later in Lemma 2.14 we will consider perturbations for different N's simultaneously)

$$\hat{h}_v^{(N)} = \begin{cases} h_v + \Delta(N) & \text{for } v \in \Lambda_N \setminus \Lambda_{N/4}, \\ h_v & \text{for } v \in \Lambda_{N/4}. \end{cases} \tag{22}$$

We then define $\hat{\mathcal{C}}^{\Lambda_N}$ similar to $\mathcal{C}^{\Lambda_N}$ but with respect to the field $\{\hat{h}_v^N : v \in \Lambda_N\}$. Further, let $\mathcal{C}_\star^{\Lambda_N} = \mathcal{C}^{\Lambda_N} \cap \hat{\mathcal{C}}^{\Lambda_N}$ (so $\mathcal{C}_\star^{\Lambda_N}$ is a version of $\mathcal{C}_*^{\Lambda_N}$, but it replaces $\tilde{\mathcal{C}}^{\Lambda_N}$ with $\hat{\mathcal{C}}^{\Lambda_N}$ in its definition).

Lemma 2.13 *Let* $\Delta(N) = (N/4)^{-\alpha(\alpha')^2}$ *and define* $\{\hat{h}_v^{(N)} : v \in \Lambda_N\}$ *as in* (22). *Then there exists* $C = C(\varepsilon) > 0$ *such that* $\mathbb{P}(o \in \mathcal{C}_\star^{\Lambda_N}) \leqslant CN^{-5}$.

Proof For $v \in \partial\Lambda_{N/2}$, let B_v be a translated copy of $\Lambda_{N/4}$ centered at v. Thus, for all $u \in B_v$ we have $\hat{h}_u^{(N)} = h_u + (N/4)^{-\alpha(\alpha')^2}$. Recall $m^\diamond_{N/4}(N/4)$ as in Lemma 2.11. By (5) and Lemma 2.11,

$$\mathbb{P}(v \in \mathcal{C}_\star^{\Lambda_N}) \leqslant m^\diamond_{N/4}(N/4) \leqslant CN^{-6}.$$

Hence, $\mathbb{P}(\partial\Lambda_{N/2} \cap \mathcal{C}_\star^{\Lambda_N} \neq \emptyset) \leqslant CN^{-5}$ by a simple union bound. Combined with Lemma 2.6 (and the simple observation that o cannot be connected to $\partial\Lambda_N$ by a path in $\mathcal{C}_\star^{\Lambda_N}$ if $\partial\Lambda_{N/2} \cap \mathcal{C}_\star^{\Lambda_N} = \emptyset$), this completes the proof of the lemma. □

Lemma 2.14 *There exists* $C = C(\varepsilon) > 0$ *such that* $m_N \leqslant CN^{-3}$.

Proof A rough intuition behind the proof is as follows: the random field in each dyadic annulus has probability close to 1 to stop the event $\{o \in \mathcal{C}^{\Lambda_N}\}$ from occurring and thus altogether we get a polynomial upper bound with large power. In order to formalize the proof, we will apply Lemma 2.13 and employ a careful analysis to justify the "independence" among different scales.

Without loss of generality, let us only consider $N = 4^n$ for some $n \geqslant 1$. For each such N, define $\{\hat{h}_v^{(N)} : v \in \Lambda_N\}$ as in (22) with $\Delta(N) = (N/4)^{-\alpha(\alpha')^2}$.

Let $E_\ell = \{o \notin \mathcal{C}_\star^{\Lambda_{4^\ell}}\}$ and $E = \cap_{0.9n \leqslant \ell \leqslant n} E_\ell$. (Note that there is no containment relation among the events E_ℓ's, since each event depends on a different perturbation.) By Lemma 2.13, we see that $\mathbb{P}(E^c) \leqslant CN^{-3}$ for some $C = C(\varepsilon) > 0$ (whose value may be adjusted later in the proof). Write $\mathfrak{A}_\ell = \Lambda_{4^\ell} \setminus \Lambda_{4^{\ell-1}}$. For $0.9n \leqslant \ell \leqslant n$, let $\mathcal{F}_\ell = \sigma(h_v : v \in \Lambda_{4^\ell})$ and write

$$h_v = (|\mathfrak{A}_\ell|)^{-1} h_{\mathfrak{A}_\ell} + g_v \quad \text{for } v \in \mathfrak{A}_\ell, \tag{23}$$

where $\{g_v : v \in \mathfrak{A}_\ell\}$ is a mean-zero Gaussian process independent of $h_{\mathfrak{A}_\ell}$ and $\{g_v : v \in \mathfrak{A}_\ell\}$ for $0.9n \leqslant \ell \leqslant n$ are mutually independent (note that g_v's are linear combinations of a Gaussian process and their means and covariances can be easily computed). Let $\mathcal{F}'_\ell$ be the σ-field which contains every event in $\mathcal{F}_\ell$ that is independent of $h_{\mathfrak{A}_\ell}$ (so in particular $\mathcal{F}_\ell \subset \mathcal{F}'_{\ell+1} \subset \mathcal{F}_{\ell+1}$). By monotonicity, there exists an interval I_ℓ measurable with respect to $\mathcal{F}'_\ell$ such that conditioned on $\mathcal{F}'_\ell$ we have $o \in \mathcal{C}^{\Lambda_{4^\ell}}$ if and only if $h_{\mathfrak{A}_\ell} \in I_\ell$. Let I'_ℓ be the maximal sub-interval of I_ℓ which shares the upper endpoint and with length $|I'_\ell| \leqslant \frac{|\mathfrak{A}_\ell| \cdot 4^{\alpha(\alpha')^2}}{4^{\alpha(\alpha')^2 \ell}}$. By our definition of E_ℓ, we see from (23) that conditioned on $\mathcal{F}'_\ell$ we have $\{o \in \mathcal{C}^{\Lambda_{4^\ell}}\} \cap E_\ell$ only if $h_{\mathfrak{A}_\ell} \in I'_\ell$. Thus, for $0.9n \leqslant \ell \leqslant n$,

$$\mathbb{P}(\{o \in \mathcal{C}^{\Lambda_{4^\ell}}\} \cap E_\ell \mid \mathcal{F}'_\ell) \leqslant \mathbb{P}(h_{\mathfrak{A}_\ell} \in I'_\ell).$$

Combined with the fact that $\mathrm{Var}(h_{\mathfrak{A}_\ell}) = \varepsilon^2 |\mathfrak{A}_\ell|$, this gives that

$$\mathbb{P}(\{o \in \mathcal{C}^{\Lambda_{4^\ell}}\} \cap E_\ell \mid \mathcal{F}'_\ell) \leqslant \frac{C}{4^{\ell(\alpha(\alpha')^2 - 1)}}.$$

Since $\{o \in \mathcal{C}^{\Lambda_{4^n}}\} \cap E = \cap_{\ell=0.9n}^{n} (\{o \in \mathcal{C}^{\Lambda_{4^\ell}}\} \cap E_\ell)$ and since $\{o \in \mathcal{C}^{\Lambda_{4^\ell}}\} \cap E_\ell$ is $\mathcal{F}_\ell$-measurable (and thus is $\mathcal{F}'_{\ell+1}$-measurable), we deduce that $\mathbb{P}(\{o \in \mathcal{C}^{\Lambda_N}\} \cap E) \leqslant CN^{-3}$. Combined with the fact that $\mathbb{P}(E^c) \leqslant CN^{-3}$, it completes the proof of the lemma. □

Proof of Theorem 2.1 Let $N_0 = N_0(\varepsilon)$ be chosen later. For $B \in \mathcal{B}(N, N_0)$, we say B is open if $\mathcal{C}^{B^{\text{large}}} \cap B \neq \emptyset$. Clearly, this percolation process satisfies the $(N, N_0, 4, p)$-condition where

$$p = \mathbb{P}(\mathcal{C}^{B^{\text{large}}} \cap B \neq \emptyset) \leqslant N_0^2 m_{N_0/2} \leqslant C N_0^{-1} \quad \text{for } C = C(\varepsilon) > 0. \tag{24}$$

(The last transition above follows from Lemma 2.14.) In addition, we note that in order for $o \in \mathcal{C}^{\Lambda_N}$, it is necessary that there exists an open lattice animal on $B \in \mathcal{B}(N, N_0)$ with size at least $\frac{N}{10N_0}$. Now, choosing N_0 sufficiently large (so that p is sufficiently small, by (24)) and applying Lemma 2.10 completes the proof. □

3 Exponential decay at positive temperatures

In this section, we prove Theorem 1.1 for the case of $T > 0$. Our proof method follows the basic framework presented in Sect. 2 for the case of $T = 0$, which applies the result in [1] in a crucial way. However, there seem to be significant additional obstacles due to the randomness of Ising measures at positive temperatures. For $T = 0$, it suffices to consider the ground state which is unique with probability 1, and thus ground states with different boundary conditions and external fields are naturally coupled together. In the case of $T > 0$, on the one hand we try to carry out our analysis with validity for all reasonable (e.g., for all monotone couplings) couplings of Ising measures whenever possible (see Sect. 3.1); on the other hand it seems necessary to construct a coupling with some desirable properties in order to apply [1] (see Sect. 3.2). Both of these require some new ideas as well as some delicate treatment.

Organization for the rest of this section is as follows. In Sect. 3.1, we verify the hypothesis in [1] via a perturbation argument and thereby prove that under any monotone coupling for Ising spins with plus/minus boundary conditions, the intrinsic distance for the induced graph on vertices with disagreements has dimension strictly larger than 1. The proof method is inspired by that of Proposition 2.2, but the implementation is largely different with new tricks involved. In Sect. 3.2, we introduce the notion of adaptive admissible coupling and a multi-scale construction of an adaptive admissible coupling is then given in Sect. 3.3.1. In Sect. 3.3.2, we then introduce another perturbation argument, using which we analyze our adaptive admissible coupling in Sect. 3.3.3 and prove a crucial estimate in Lemma 3.17. In Sect. 3.4, we provide the Proof of Theorem 1.1 for $T > 0$, which requires to employ an admissible coupling such that the disagreement percolates to the boundary.

3.1 Intrinsic distance on disagreements via a perturbation argument

For any $A \subset \mathbb{Z}^2$, we continue to denote by $d_A(\cdot, \cdot)$ the intrinsic distance on A, i.e., the graph distance on the induced subgraph on A. Let $\sigma^{\Lambda_N,\pm}$ be spins sampled according to $\mu^{\Lambda_N,\pm}$. We will continue to use repeatedly the standard monotonicity properties of the Ising model with respect to external fields and boundary conditions (c.f. [3, Section 2.2] for detailed discussions). Let π be a monotone coupling of $\mu^{\Lambda_N,\pm}$ (that is, under π we have $\sigma^{\Lambda_N,+} \geqslant \sigma^{\Lambda_N,-}$) and let

$$\mathcal{C}^{\Lambda_N} = \mathcal{C}^{\Lambda_N,\pi} = \{v \in \Lambda_N : \sigma_v^{\Lambda_N,+} > \sigma_v^{\Lambda_N,-}\}. \tag{25}$$

(Note that π depends on the random field h.) In addition, denote by $\mathbb{P} \otimes \pi$ the joint measure of the external fields and the spin configurations (similar notations also apply below). The following proposition is the major goal of this section.

Proposition 3.1 *There exist $\alpha = \alpha(\varepsilon, \beta) > 1$, $\kappa = \kappa(\varepsilon, \beta) > 0$ such that the following holds. For all $0 < c \leqslant 1$, there exists $N_0 = N_0(\varepsilon, \beta, c)$ such that for all $N \geqslant N_0$ and $1 \leqslant N_1 \leqslant N_2 \leqslant N/2$ with $N_2 - N_1 \geqslant N^c$ the following holds for all monotone coupling π of $\mu^{\Lambda_N, \pm}$:*

$$\mathbb{P} \otimes \pi(d_{\mathcal{C}^{\Lambda_N}}(\partial \Lambda_{N_1}, \partial \Lambda_{N_2}) \leqslant (N_2 - N_1)^{\alpha}) \leqslant \kappa^{-1} e^{-N^{\kappa c}}. \tag{26}$$

Remark 3.2 (1) The preceding proposition is analogous to Proposition 2.2. In the present case, it is crucial that the result holds for all monotone couplings (note that the intrinsic distance may depend on the coupling), so that we can apply it to couplings which we construct later.

(2) In Proposition 3.1, we introduce parameters N_1, N_2 (as opposed to $N_1 = N/4$ and $N_2 = N/2$ in Proposition 2.2) for convenience of later applications. The condition that $N_2 - N_1 \geqslant N^c$ is just to ensure that the decay in probability absorbs the number of choices for starting and ending points of the shortest path. This slight extension does not introduce complication to the proof.

The Proof of Proposition 3.1 again crucially relies on the result of [1]. In order to apply [1], the following lemma (analogous to Lemma 2.8) is a key ingredient. For any annulus $\mathcal{A}$ and $\mathcal{C} \subset \mathbb{Z}^2$, we continue to denote by $\mathrm{Cross}_{\mathrm{hard}}(\mathcal{A}, \mathcal{C})$ the event that there is a contour in $\mathcal{C}$ which separates the inner and outer boundaries of $\mathcal{A}$. Let

$$\mathcal{E}^{\pm} = \mathcal{E}^{\pm}_N = \mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \{v \in \Lambda_N : \sigma_v^{\Lambda_N, \pm} = \pm 1\}). \tag{27}$$

Lemma 3.3 *There exists $\delta = \delta(\varepsilon, \beta) > 0$ such that for all $N \geqslant 32$*

$$\min\{\mathbb{P} \otimes \mu^{\Lambda_N,+}(\mathcal{E}^+), \\ \mathbb{P}(\sum_{v \in \Lambda_{N/8}} (\langle \sigma_v^{\Lambda_N,+} \rangle_{\mu^{\Lambda_N,+}} - \langle \sigma_v^{\Lambda_N,-} \rangle_{\mu^{\Lambda_N,-}}) > 10^{-3} N)\} \leqslant 1 - \delta.$$

In particular, $\mathbb{P} \otimes \pi(\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{C}^{\Lambda_N})) \leqslant 1 - \delta$ for all monotone coupling π of $\mu^{\Lambda_N, \pm}$.

Remark 3.4 By Lemma 3.3, either of the following holds: (1) with positive probability the plus-spins with respect to the plus boundary condition does not separate the boundaries of an annulus (this is a stronger than what was proved in Case 1 in the Proof of Lemma 2.8); (2) with positive probability the expected number of disagreements (averaged over the Ising measures) is small (this corresponds to Case 2 in the Proof of Lemma 2.8). Assuming either property, we are able to derive a uniform bound on crossing probabilities for disagreements under any monotone coupling.

After establishing exponential decay, then it is clear that Property (2) holds. In addition, we know that with overwhelming probability away from the boundary the spin configurations with plus and minus boundary conditions agree with each other. Therefore, by symmetry and planar duality we see that Property (1) also holds.

3.1.1 A perturbative analysis

Before proving Lemma 3.3, we need some preparational work on a certain perturbative analysis. This is analogous to Lemma 2.5, which has been applied twice in the case of $T = 0$: in the Proof of Lemma 2.8 and the Proof of Lemma 2.11. For $T > 0$, it is more complicated and thus we provide two separate versions of perturbative analysis, both of which are proved via keeping track of the free energy. The first version is presented in Lemma 3.5 in the present section (for the application in Lemma 3.3), and the second version is presented in Sect. 3.3.2 (for the application in Lemma 3.17).

For any set $\Lambda \subset \mathbb{Z}^2$ and a configuration $\tau \in \{-1, 1\}^{\partial\Lambda}$, analogous to (1) we can define the Hamiltonian on Λ with boundary condition τ and external field $\{h_v\}$ by:

$$H^{\Lambda,\tau}(\sigma) = -\Big(\sum_{u\sim v, u,v\in\Lambda} \sigma_u\sigma_v + \sum_{u\sim v, u\in\Lambda, v\in\partial\Lambda} \sigma_u\tau_v + \sum_{u\in\Lambda} \sigma_u h_u \Big) \quad (28)$$

for $\sigma \in \{-1, 1\}^\Lambda$. We can then analogously define the Ising measure $\mu^{\Lambda,\tau}$ by assigning probability to $\sigma \in \{-1, 1\}^\Lambda$ proportional to $e^{-\beta H^{\Lambda,\tau}(\sigma)}$. In addition, we define the corresponding log-partition-function (it is the negative of the free energy; in our analysis, it seems cleaner to work with the log-partition-function so not to be confused by the negative sign)

$$F^{\Lambda,\tau} = \frac{1}{\beta} \log \Big(\sum_{\sigma\in\{-1,1\}^\Lambda} e^{-\beta H^{\Lambda,\tau}(\sigma)} \Big). \quad (29)$$

For simplicity, we will only consider $N = 2^n$ for $n \geqslant 10$. For $\Delta > 0$, $\Delta' \geqslant 0$ and $0 \leqslant t \leqslant 1$, we will consider the following perturbed field in this section (which is increasing in t):

$$h_v^{(t)} = h_v^{(t,N)} = \begin{cases} h_v + \Delta' & \text{for } v \in \Lambda_N \setminus \Lambda_{N/8}, \\ h_v + t\Delta & \text{for } v \in \Lambda_{N/8}. \end{cases} \tag{30}$$

(We draw the reader's attention to that t appeared in the definition of $h_v^{(t)}$ only for $v \in \Lambda_{N/8}$, and that $h^{(0)} \neq h$ if $\Delta' > 0$. The perturbation in (30) is more subtle than that in (9), for the reason that we wish to take advantage of (41) below later with a judicious choice of Δ'.) Let $\mu^{\Lambda_N,\pm,t}$ be Ising measures with plus/minus boundary conditions and external field $\{h_v^{(t)} : v \in \Lambda_N\}$. In addition, let $H^{\Lambda_N,\pm,t}$ be the corresponding Hamiltonians, let $F^{\Lambda_N,\pm,t}$ be the corresponding log-partition-functions, and let $\sigma^{\Lambda_N,\pm,t}$ be spin configurations sampled according to $\mu^{\Lambda_N,\pm,t}$.

For notation convenience, for any set $\Gamma \subset \mathbb{Z}^2$, let S_Γ be the collection of vertices which are not in Γ and are separated by Γ from ∞ on $\mathbb{Z}^2$ (i.e., the collection of vertices that are enclosed by Γ).

Let $S \subset \Lambda_N$ be a subset which contains $\Lambda_{N/8}$ and let $\Gamma = \partial S$ (thus we have $S \subset S_\Gamma$).

For any $\tau \in \{-1, 1\}^\Gamma$, we denote by $\mu^{S,\tau,t}$ the Ising measure on S with boundary condition τ and external field $\{h_v^{(t)} : v \in S\}$. In addition, let $H^{S,\tau,t}$ be the Hamiltonian for the corresponding Ising spin, and let $F^{S,\tau,t}$ be the corresponding log-partition-function. Also, we let $\sigma^{S,\tau,t}$ be the spin configuration sampled according to $\mu^{S,\tau,t}$. For later applications, it would be useful to consider the log-partition-function restricted to a subset of configurations. To this end, we define

$$F_\Omega^{S,\tau,t} = \frac{1}{\beta} \log \Big(\sum_{\sigma \in \Omega} e^{-\beta H^{S,\tau,t}(\sigma)} \Big) \quad \text{for } \Omega \subset \{-1, 1\}^S. \tag{31}$$

In addition, for any measure $\mu^{S,\tau,t}$, we define $\mu_\Omega^{S,\tau,t}$ to be a measure such that

$$\mu_\Omega^{S,\tau,t}(\sigma) = (\mu^{S,\tau,t}(\Omega))^{-1} \mu^{S,\tau,t}(\sigma) \quad \text{for } \sigma \in \Omega.$$

(We draw readers' attention to that $\mu^{S,\tau,t}(\Omega)$ is the total measure of Ω under $\mu^{S,\tau,t}$ and thus is a number, and that $\mu_\Omega^{S,\tau,t}$ is the measure $\mu^{S,\tau,t}$ conditioned on the occurrence of Ω.) For convenience, we let $\sigma_\Omega^{S,\tau,t}$ be the spin configuration sampled according to $\mu_\Omega^{S,\tau,t}$. Further, define (note that below we sum over $v \in \Lambda_{N/32}$ as opposed to $v \in S$)

$$m_\Omega^{S,\tau,t} = \sum_{v \in \Lambda_{N/32}} \langle \sigma_{\Omega,v}^{S,\tau,t} \rangle_{\mu_\Omega^{S,\tau,t}}. \tag{32}$$

For notation convenience, we write $m^{S,\tau,t} = m_{\Omega}^{S,\tau,t}$ if $\Omega = \{-1,1\}^S$. We say $\Omega \subset \{-1,1\}^S$ is an increasing set if $\sigma \in \Omega$ implies that $\sigma' \in \Omega$ provided $\sigma' \geqslant \sigma$, and we say Ω is a decreasing set if Ω^c is an increasing set. In what follows, we consider $\tau^+, \tau^- \in \{-1,1\}^\Gamma$ such that $\tau^+ \geqslant \tau^-$.

Lemma 3.5 *Quench on the external field $\{h_v\}$. We have that for any increasing set $\Omega^+ \subset \{-1,1\}^S$ and any decreasing set $\Omega^- \subset \{-1,1\}^S$*

$$\begin{aligned}\Delta \int_0^1 (m_{\Omega^+}^{S,\tau^+,t} - m_{\Omega^-}^{S,\tau^-,t})dt \\ \leqslant 8 \sum_{v\in\Gamma} (\tau_v^+ - \tau_v^-) - \frac{1}{\beta}\big(\log \mu^{S,\tau^+,0}(\Omega^+) + \log \mu^{S,\tau^-,1}(\Omega^-)\big).\end{aligned}$$

Proof The proof is done via keeping track of the change on the difference of log-partition-functions with respect to different boundary conditions when we perturb the external field. In **Step 1**, we bound such difference from above by the number of disagreements on boundary conditions; in **Step 2** we bound such difference from below by the expected number of disagreements, with a caveat that we use the notion of "restricted" log-partition-functions as in (31); in **Step 3**, we address the caveat by linking the two notions of log-partition-functions.

Step 1. We will prove (below the equality is obvious since $\tau^+ \geqslant \tau^-$)

$$\begin{aligned}(F^{S,\tau^+,1} - F^{S,\tau^-,1}) - (F^{S,\tau^+,0} - F^{S,\tau^-,0}) \\ \leqslant 16 \cdot \#\{v \in \Gamma : \tau_v^+ \neq \tau_v^-\} = 8 \sum_{v\in\Gamma} (\tau_v^+ - \tau_v^-). \end{aligned} \tag{33}$$

(Here we use $\#A$ to denote the cardinality of A for a finite set A. We switch from the more compact notation $|A|$ to $\#A$ in this section, as we wish to avoid somewhat awkward notation when $|$ is followed by another $|$ which means "conditioned on".) Since each vertex has 4 neighbors in $\mathbb{Z}^2$, a straightforward computation gives that

$$\begin{aligned}F^{S,\tau^+,1} - F^{S,\tau^-,1} &= \frac{1}{\beta} \log \frac{\sum_\sigma e^{-\beta H^{S,\tau^+,1}(\sigma)}}{\sum_\sigma e^{-\beta H^{S,\tau^-,1}(\sigma)}} \\ &\leqslant \frac{1}{\beta} \log e^{8\beta \cdot \#\{v\in\Gamma : \tau_v^+ \neq \tau_v^-\}} \leqslant 8 \cdot \#\{v \in \Gamma : \tau_v^+ \neq \tau_v^-\}.\end{aligned}$$

Similarly, we have that $F^{S,\tau^+,0} - F^{S,\tau^-,0} \geqslant -8 \cdot \#\{v \in \Gamma : \tau_v^+ \neq \tau_v^-\}$. This proves (33).

Step 2. We will prove

$$(F_{\Omega^+}^{S,\tau^+,1} - F_{\Omega^-}^{S,\tau^-,1}) - (F_{\Omega^+}^{S,\tau^+,0} - F_{\Omega^-}^{S,\tau^-,0})$$

$$\geqslant \Delta \int_0^1 (m_{\Omega^+}^{S,\tau^+,t} - m_{\Omega^-}^{S,\tau^-,t})dt. \tag{34}$$

We write

$$\begin{aligned}(F_{\Omega^+}^{S,\tau^+,1} - F_{\Omega^-}^{S,\tau^-,1}) &- (F_{\Omega^+}^{S,\tau^+,0} - F_{\Omega^-}^{S,\tau^-,0}) \\ &= (F_{\Omega^+}^{S,\tau^+,1} - F_{\Omega^+}^{S,\tau^+,0}) - (F_{\Omega^-}^{S,\tau^-,1} - F_{\Omega^-}^{S,\tau^-,0}).\end{aligned} \tag{35}$$

Thus, we get that

$$F_{\Omega^\pm}^{S,\tau^\pm,1} - F_{\Omega^\pm}^{S,\tau^\pm,0} = \int_0^1 \frac{dF_{\Omega^\pm}^{S,\tau^\pm,t}}{dt} dt. \tag{36}$$

Since $\frac{dF_{\Omega^\pm}^{S,\tau^\pm,t}}{dt} = \sum_{v\in\Lambda_{N/8}} \Delta\langle\sigma_{\Omega^\pm,v}^{S,\tau^\pm,t}\rangle_{\mu_{\Omega^\pm}^{S,\tau^\pm,t}}$, we see

$$\begin{aligned}\frac{dF_{\Omega^+}^{S,\tau^+,t}}{dt} - \frac{dF_{\Omega^-}^{S,\tau^-,t}}{dt} &\geqslant \sum_{v\in\Lambda_{N/32}} \Delta(\langle\sigma_{\Omega^+,v}^{S,\tau^+,t}\rangle_{\mu_{\Omega^+}^{S,\tau^+,t}} - \langle\sigma_{\Omega^-,v}^{S,\tau^-,t}\rangle_{\mu_{\Omega^-}^{S,\tau^-,t}}) \\ &= \Delta m_{\Omega^+}^{S,\tau^+,t} - \Delta m_{\Omega^-}^{S,\tau^-,t},\end{aligned}$$

where the inequality follows from the fact that

$$\begin{aligned}\langle\sigma_{\Omega^+,v}^{S,\tau^+,t}\rangle_{\mu_{\Omega^+}^{S,\tau^+,t}} \geqslant \langle\sigma_v^{S,\tau^+,t}\rangle_{\mu^{S,\tau^+,t}} \geqslant \langle\sigma_v^{S,\tau^-,t}\rangle_{\mu^{S,\tau^-,t}} \geqslant \langle\sigma_{\Omega^-,v}^{S,\tau^-,t}\rangle_{\mu_{\Omega^-}^{S,\tau^-,t}} \\ \text{for all } v \in S,\end{aligned}$$

In the preceding display, the first and the third inequalities follow from FKG inequality [13] and the second inequality follows from monotonicity. Combined with (36) and (35), it yields (34).

Step 3. From definitions as in (29) and (31), we see that

$$F^{S,\tau^+,1} - F_{\Omega^+}^{S,\tau^+,1} = -\frac{1}{\beta} \log \mu^{S,\tau^+,1}(\Omega^+), \tag{37}$$

and similar equalities hold for other combinations of boundary conditions, external fields and $\Omega^\pm$.

Combining (33), (34) and (37), we complete the proof of the lemma. □

3.1.2 A lower bound on the intrinsic distance

Denote by $\mathcal{V}^{\sigma,\pm} = \{v \in S : \sigma_v = \pm 1\}$ for $S \subset \Lambda_N$ and $\sigma \in \{-1, 1\}^S$. For any $S \supset \Lambda_{N/8}$, define

$$\Omega^{\pm} = \Omega^{\pm}(S) = \{\sigma \in \{-1, 1\}^S : \mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{V}^{\sigma,\pm}) \text{ occurs } \}. \tag{38}$$

We see that Ω^+ is an increasing set and Ω^- is a decreasing set. For $A \subset \Lambda \subset \mathbb{Z}^2$ and $\sigma \in \{-1, 1\}^\Lambda$, we denote by σ_A the restriction of σ on A. Let $r > 0$ be a constant chosen later. Recall (30). Let $\Delta = \frac{10^{10} r^8}{N(\beta \wedge 1)}$ and $\Delta' = t^* \Delta$ for $0 \leqslant t^* \leqslant 1$ to be chosen.

Lemma 3.6 *For any $p, r > 0$, there exists $c = c(\varepsilon, p, r, \beta) > 0$ such that for any event E_N with $\mathbb{P}(\{h_v^{(t)} : v \in \Lambda_N\} \in E_N) \geqslant p$ for some $0 \leqslant t, t^* \leqslant 1$, we have that $\mathbb{P}(\{h_v : v \in \Lambda_N\} \in E_N) \geqslant c$.*

Proof The proof is an adaption of Lemma 2.7 except for minimal notation change, and thus we omit further details. □

Proof of Lemma 3.3 The proof shares similarity with that of Lemma 2.8, but the present proof is substantially more involved. We first provide a heuristic outline of the proof, and we will not be precise on notations or unimportant constants in this informal description. The statement will follow immediately if the probability for existence of a plus contour with respect to plus boundary condition is strictly less than 1, and thus we suppose otherwise (formally, we suppose (39) below). We wish to compare the number of disagreements in $\Lambda_{N/32}$ with that in $\mathcal{A}_{N/2}$. To this end, it will be useful to consider the "enhanced" disagreements in $\Lambda_{N/32}$ (that is, when we pose plus and minus boundary conditions on $\partial\Lambda_{N/8}$ instead of $\partial\Lambda_N$; the word "enhanced" is chosen because by monotonicity the enhanced disagreements stochastically dominate the original disagreements). We now compare the enhanced disagreements in $\Lambda_{N/32}$ and disagreements in $\mathcal{A}_{N/2}$ in both directions.

- The "$\leqslant$" direction (**Step 1** below): This is where plus (minus) contours come into play. Conditioned on existence of plus and minus contours, the disagreements in $\Lambda_{N/32}$ stochastically dominate the enhanced disagreements. In addition, by Lemma 3.5, the number of disagreements in $\Lambda_{N/32}$ is upper bounded by that in $\mathcal{A}_{N/2}$ (up to an additive term that is related to the probability of existence of plus/minus contours, which we will address later). Altogether, we get that the number of enhanced disagreements in $\Lambda_{N/32}$ is upper bounded by the number of disagreements in $\mathcal{A}_{N/2}$ (see (49)).

- The "$\geqslant$" direction (**Step 2** below): The set of disagreements in $\mathcal{A}_{N/2}$ is dominated by a union of constant copies of enhanced disagreements in $\Lambda_{N/32}$, where the number of disagreements in all these copies are independent of the enhanced disagreements in $\Lambda_{N/32}$ (but not of each other). This implies that with positive probability, the number of enhanced disagreements in $\Lambda_{N/32}$ is larger (up to a constant factor) than the number of disagreements in $\mathcal{A}_{N/2}$ (see (53)).

Now, if we choose the constants appropriately, we will see that the preceding two scenarios will occur simultaneously with positive probability, which yield bounds in two directions that "almost" contradict each other. These events can only happen concurrently if the logarithmic term we ignored earlier (which becomes $\frac{N}{2\beta}$ in (49)) plays a significant role. But this can happen only when the typical number of enhanced disagreements is at most of order N, in which case an application of Markov's inequality (see (45)) yields the desired lemma.

We next carry out the proof formally, where we slightly shuffle the order of arguments: we first show that if the typical number of enhanced disagreements is at most of order N (see (42)), then the lemma holds. Next, we prove (42) (which is the main challenge) by contradiction, via the aforementioned two directional comparisons.

For convenience of notation, write

$$\mathcal{E}^{\pm,t} = \mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{V}^{\sigma^{\Lambda_N,\pm,t},\pm}).$$

We suppose that

$$\min_{0\leqslant t\leqslant 1} \{\mathbb{P} \otimes \mu^{\Lambda_N,+,t}(\mathcal{E}^{+,t}), \mathbb{P} \otimes \mu^{\Lambda_N,-,t}(\mathcal{E}^{-,t})\} \geqslant 1 - r^{-4}10^{-10}. \quad (39)$$

Otherwise Lemma 3.3 follows from Lemma 3.6 (since under any monotone coupling we have $\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8} \setminus \Lambda_{N/32}, \mathcal{C}^{\Lambda_N}) \subset \mathcal{E}_N^+ \cap \mathcal{E}_N^-$, where $\mathcal{E}_N^\pm$ is defined in (27)). We remark that by monotonicity the preceding inequality is equivalent to $\min\{\mathbb{P} \otimes \mu^{\Lambda_N,+,0}(\mathcal{E}^{+,0}), \mathbb{P} \otimes \mu^{\Lambda_N,-,1}(\mathcal{E}^{-,1})\} \geqslant 1 - r^{-4}10^{-10}$.

Let $\mathcal{E}^\star = \{\mu^{\Lambda_N,+,0}(\mathcal{E}^{+,0}) \geqslant 99/100\} \cap \{\mu^{\Lambda_N,-,1}(\mathcal{E}^{-,1}) \geqslant 99/100\}$ be an event measurable with respect to the Gaussian field. By (39), we see that

$$\mathbb{P}(\mathcal{E}^\star) \geqslant 1 - 10^{-2}r^{-4}. \quad (40)$$

Let $t^* \in [0, 1]$ be such that

$$\inf\{\theta : \mathbb{P}(m^{\Lambda_{N/8},+,t^*} - m^{\Lambda_{N/8},-,t^*} \geqslant \theta) \leqslant 1/2r\} = \theta^*, \quad (41)$$

where $\theta^* = \min_{0\leqslant t\leqslant 1}\inf\{\theta : \mathbb{P}(m^{\Lambda_{N/8},+,t} - m^{\Lambda_{N/8},-,t} \geqslant \theta) \leqslant 1/2r\}$. We claim that

$$\theta^* \leqslant 10^{-3} r^{-1} N. \tag{42}$$

We first show that (42) implies the lemma. For any box A, let A^{Big} be the concentric box of A with side length 4 times that of A. Let r be a large enough constant so that we can write $\Lambda_{N/8} = \cup_{i=1}^{r} A_i$, where A_i is a copy of $\Lambda_{N/32}$ and A_i's are disjoint such that $A_i^{\mathrm{Big}} \subset \Lambda_N$ for $1 \leqslant i \leqslant r$. By monotonicity, we see that for each $1 \leqslant i \leqslant r$

$$\begin{aligned}
&\mathbb{P}(\sum_{v\in A_i}(\langle\sigma_v^{\Lambda_N,+,t^*}\rangle_{\mu^{\Lambda_N,+,t^*}} - \langle\sigma_v^{\Lambda_N,-,t^*}\rangle_{\mu^{\Lambda_N,-,t^*}}) > \theta^*)\\
&\quad\leqslant \mathbb{P}(\sum_{v\in A_i}(\langle\sigma_v^{A_i^{\mathrm{Big}},+,t^*}\rangle_{\mu^{A_i^{\mathrm{Big}},+,t^*}} - \langle\sigma_v^{A_i^{\mathrm{Big}},-,t^*}\rangle_{\mu^{A_i^{\mathrm{Big}},-,t^*}}) > \theta^*) \leqslant (2r)^{-1},
\end{aligned}$$

where the last inequality holds due to our choice of t^* as in (41) and $\Delta' = t^*\Delta$ (thus $h_v^{(t^*)} = h_v + \Delta'$ for $v \in \Lambda_N$). Hence, a simple union bound gives that

$$\mathbb{P}(\sum_{v\in\Lambda_{N/8}}(\langle\sigma_v^{\Lambda_N,+,t^*}\rangle_{\mu^{\Lambda_N,+,t^*}} - \langle\sigma_v^{\Lambda_N,-,t^*}\rangle_{\mu^{\Lambda_N,-,t^*}}) \leqslant r\theta^*) \geqslant \frac{1}{2}. \tag{43}$$

By Lemma 3.6, we get that

$$\begin{aligned}
&\mathbb{P}(\sum_{v\in\Lambda_{N/8}}(\langle\sigma_v^{\Lambda_N,+}\rangle_{\mu^{\Lambda_N,+}} - \langle\sigma_v^{\Lambda_N,-}\rangle_{\mu^{\Lambda_N,-}}) > r\theta^*)\\
&\quad\leqslant 1-\delta \quad \text{for } \delta = \delta(\varepsilon,\beta,r) > 0.
\end{aligned} \tag{44}$$

Note that $2\langle\#(\mathcal{C}^{\Lambda_N} \cap \Lambda_{N/8})\rangle_\pi = \sum_{v\in\Lambda_{N/8}}(\langle\sigma_v^{\Lambda_N,+}\rangle_{\mu^{\Lambda_N,+}} - \langle\sigma_v^{\Lambda_N,-}\rangle_{\mu^{\Lambda_N,-}})$ on each instance of the Gaussian field for any monotone coupling π of $\mu^{\Lambda_N,\pm}$. Therefore, on each instance of Gaussian field (which occurs with probability at least δ) such that $\sum_{v\in\Lambda_{N/8}}(\langle\sigma_v^{\Lambda_N,+}\rangle_{\mu^{\Lambda_N,+}} - \langle\sigma_v^{\Lambda_N,-}\rangle_{\mu^{\Lambda_N,-}}) \leqslant r\theta^*$, we apply Markov's inequality and get that

$$\begin{aligned}
&\pi(\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8}\setminus\Lambda_{N/32}, \mathcal{C}^{\Lambda_N}))\\
&\quad\leqslant \pi(\#(\mathcal{C}^{\Lambda_N}\cap\Lambda_{N/8}) \geqslant \tfrac{N}{32}) \leqslant \frac{\theta^* r}{N/32} \leqslant \frac{1}{2},
\end{aligned} \tag{45}$$

where the last inequality follows from (42). This implies that $\mathbb{P}\otimes\pi(\mathrm{Cross}_{\mathrm{hard}}(\Lambda_{N/8}\setminus\Lambda_{N/32}, \mathcal{C}^{\Lambda_N})) \leqslant 1-\delta/2$, completing the Proof of Lemma 3.3 (combined with (44)).

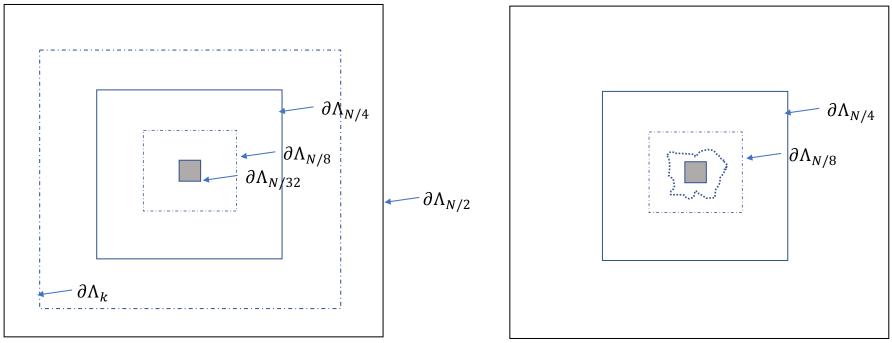

Fig. 3 Illustrations for geometric setup in Step 1 of Lemma 3.3. The picture on the left illustrates the setup for derivation of (47), where we bound disagreements in the grey square by disagreements on $\partial\Lambda_k$ (the larger dot-line boundary). The picture on the right illustrates the setup for derivation of (48): by FKG conditioned on plus (respectively minus) contour (drawn in dots in the picture) the magnetization on the grey box is pushed up (respectively down); this allows us to compare the disagreements and enhanced disagreements

It remains to prove (42). Suppose that (42) does not hold. We will derive a contradiction, using the following two steps.

Step 1. We refer to Fig. 3 for an illustration of geometric setup in this step. Fix $N/4 \leqslant k \leqslant N/2$. Write $S = \Lambda_k$ and $\Gamma = \partial S$. We first quench on the Gaussian field and also condition on

$$(\sigma^{\Lambda_N,+,1})_\Gamma=\tau^+ \text{ and } (\sigma^{\Lambda_N,-,0})_\Gamma=\tau^- \text{ where } \tau^\pm \in \{-1,1\}^\Gamma \text{ and } \tau^+ \geqslant \tau^-. \tag{46}$$

Applying Lemma 3.5, we get that (recall $\Omega^\pm = \Omega^\pm(S)$ as in (38))

$$\Delta \int_0^1 (m_{\Omega^+}^{S,\tau^+,t} - m_{\Omega^-}^{S,\tau^-,t})dt$$
$$\leqslant 8\sum_{v\in\Gamma}(\tau_v^+ - \tau_v^-) - \frac{1}{\beta}\Big(\log\mu^{S,\tau^+,0}(\Omega^+) + \log\mu^{S,\tau^-,1}(\Omega^-)\Big). \tag{47}$$

Conditioned on $\sigma^{S,\tau^+,t} \in \Omega^+$, let $\mathfrak{C} \subset \mathcal{V}^{\sigma^{S,\tau^+,t},+} \cap (\Lambda_{N/8} \setminus \Lambda_{N/32})$ be the outmost contour which surrounds $\Lambda_{N/32}$. Note that $\mathfrak{C} = \Gamma'$ is measurable with respect to $\{\sigma_v^{S,\tau^+,t} : v \in S_{\Gamma'}^c\}$. Thus, by monotonicity of Ising model we see that $(\sigma^{S,\tau^+,t})_{\Lambda_{N/32}}$ conditioned on $\mathfrak{C} = \Gamma'$ stochastically dominates $(\sigma^{\Lambda_{N/8},+,t})_{\Lambda_{N/32}}$. A similar analysis applies to $(\sigma^{S,\tau^-,t})_{\Lambda_{N/32}}$. Combined with (47), it yields that

$$\Delta \int_0^1 (m^{\Lambda_{N/8},+,t} - m^{\Lambda_{N/8},-,t})dt$$
$$\leqslant 8 \sum_{v\in\Gamma} (\tau_v^+ - \tau_v^-) - \frac{1}{\beta}\big(\log \mu^{S,\tau^+,0}(\Omega^+) + \log \mu^{S,\tau^-,1}(\Omega^-)\big). \tag{48}$$

Define $\mathcal{E}_{\Gamma,+} = \{\tau^+ : \mu^{S,\tau^+,0}(\Omega^+) \geqslant 3/4\}$ and $\mathcal{E}_{\Gamma,-} = \{\tau^- : \mu^{S,\tau^-,1}(\Omega^-) \geqslant 3/4\}$. Thus,

$$\begin{aligned}
&\mu^{\Lambda_N,+,0}(\mathcal{E}^{+,0}) \\
&\quad = \mu^{\Lambda_N,+,0}(\mathcal{E}^{+,0} \mid (\sigma^{\Lambda_N,+,0})_\Gamma \in \mathcal{E}_{\Gamma,+})\mu^{\Lambda_N,+,0}((\sigma^{\Lambda_N,+,0})_\Gamma \in \mathcal{E}_{\Gamma,+}) \\
&\qquad + \mu^{\Lambda_N,+,0}(\mathcal{E}^{+,0} \mid (\sigma^{\Lambda_N,+,0})_\Gamma \notin \mathcal{E}_{\Gamma,+})\mu^{\Lambda_N,+,0}((\sigma^{\Lambda_N,+,0})_\Gamma \notin \mathcal{E}_{\Gamma,+}) \\
&\quad \leqslant \mu^{\Lambda_N,+,0}((\sigma^{\Lambda_N,+,0})_\Gamma \in \mathcal{E}_{\Gamma,+}) + \tfrac{3}{4}\mu^{\Lambda_N,+,0}((\sigma^{\Lambda_N,+,0})_\Gamma \notin \mathcal{E}_{\Gamma,+}).
\end{aligned}$$

Since $\mu^{\Lambda_N,+,0}(\mathcal{E}^{+,0}) \geqslant 99/100$ on $\mathcal{E}^\star$, it gives that $\mu^{\Lambda_N,+,0}((\sigma^{\Lambda_N,+,0})_\Gamma \in \mathcal{E}_{\Gamma,+}) \geqslant 3/4$ and thus by monotonicity $\mu^{\Lambda_N,+,1}((\sigma^{\Lambda_N,+,1})_\Gamma \in \mathcal{E}_{\Gamma,+}) \geqslant 3/4$ (note $\mathcal{E}_{\Gamma,+}$ is an increasing set). Similarly, we get $\mu^{\Lambda_N,-,0}((\sigma^{\Lambda_N,-,0})_\Gamma \in \mathcal{E}_{\Gamma,-}) \geqslant 3/4$ on $\mathcal{E}^\star$. Consider an arbitrary monotone coupling π_Γ of $\mu^{\Lambda_N,+,1}$ and $\mu^{\Lambda_N,-,0}$ restricted to Γ. Then we see that on $\mathcal{E}^\star$

$$\pi_\Gamma(\mathcal{E}_{\Gamma,+,-}) \geqslant \tfrac{3}{4}+\tfrac{3}{4}-1 = \tfrac{1}{2} \quad \text{where } \mathcal{E}_{\Gamma,+,-}=\{(\sigma^{\Lambda_N,+,1})_\Gamma\in\mathcal{E}_{\Gamma,+},\, (\sigma^{\Lambda_N,-,0})_\Gamma\in\mathcal{E}_{\Gamma,-}\}.$$

Averaging (48) over the conditioning of (46) but restricted to the event $\mathcal{E}_{\Gamma,+,-}$, we get that on $\mathcal{E}^\star$

$$\frac{\Delta}{2}\int_0^1 (m^{\Lambda_{N/8},+,t} - m^{\Lambda_{N/8},-,t})dt \leqslant 8\sum_{v\in\Gamma}\langle(\sigma_v^{\Lambda_N,+,1} - \sigma_v^{\Lambda_N,-,0})\mathbf{1}_{\mathcal{E}_{\Gamma,+,-}}\rangle_{\pi_\Gamma} + 2/\beta.$$

Since π_Γ is a monotone coupling, we thus obtain that on $\mathcal{E}^\star$

$$\begin{aligned}
&\frac{\Delta}{2}\int_0^1 (m^{\Lambda_{N/8},+,t} - m^{\Lambda_{N/8},-,t})dt \\
&\quad \leqslant 8\sum_{v\in\Gamma}\langle\sigma_v^{\Lambda_N,+,1} - \sigma_v^{\Lambda_N,-,0}\rangle_{\pi_\Gamma} + 2/\beta \\
&\quad = 8\sum_{v\in\Gamma}(\langle\sigma_v^{\Lambda_N,+,1}\rangle_{\mu^{\Lambda_N,+,1}} - \langle\sigma_v^{\Lambda_N,-,0}\rangle_{\mu^{\Lambda_N,-,0}}) + 2/\beta.
\end{aligned}$$

Summing over $N/4 \leqslant k \leqslant N/2$, we deduce that on $\mathcal{E}^\star$

$$8 \sum_{v \in \mathcal{A}_{N/2}} (\langle \sigma_v^{\Lambda_N,+,1} \rangle_{\mu^{\Lambda_N,+,1}} - \langle \sigma_v^{\Lambda_N,-,0} \rangle_{\mu^{\Lambda_N,-,0}}) + \frac{N}{2\beta}$$
$$\geqslant \frac{N\Delta}{8} \int_0^1 (m^{\Lambda_{N/8},+,t} - m^{\Lambda_{N/8},-,t}) dt. \tag{49}$$

Step 2. For $N \geqslant 2$, recall that $\mathcal{A}_N = \Lambda_N \setminus \Lambda_{N/2}$ is an annulus. Adjust the value of r if necessary so that we can write $\mathcal{A}_{N/2} = \cup_{i=1}^r A_i$, where A_i is a copy of $\Lambda_{N/32}$ and A_i's are disjoint such that

$$A_i^{\mathrm{Big}} \subset \Lambda_N \setminus \Lambda_{N/8} \quad \text{for all } 1 \leqslant i \leqslant r. \tag{50}$$

(The geometric setup here is similar to that in the Proof of Lemma 2.8; see the left picture of Fig. 1 for an illustration.) By monotonicity, we see that for each $1 \leqslant i \leqslant r$

$$\mathbb{P}(\sum_{v \in A_i} (\langle \sigma_v^{\Lambda_N,+,1} \rangle_{\mu^{\Lambda_N,+,1}} - \langle \sigma_v^{\Lambda_N,-,0} \rangle_{\mu^{\Lambda_N,-,0}}) > \theta^*)$$
$$\leqslant \mathbb{P}(\sum_{v \in A_i} (\langle \sigma_v^{A_i^{\mathrm{Big}},+,1} \rangle_{\mu^{A_i^{\mathrm{Big}},+,1}} - \langle \sigma_v^{A_i^{\mathrm{Big}},-,0} \rangle_{\mu^{A_i^{\mathrm{Big}},-,0}}) > \theta^*)$$
$$= \mathbb{P}(m^{\Lambda_{N/8},+,t^*} - m^{\Lambda_{N/8},-,t^*} > \theta^*) \leqslant 1/2r,$$

where the equality holds due to (50) and $\Delta' = t^*\Delta$ (note that $h_v^{(t)} = h_v + \Delta'$ for $v \in \Lambda_N \setminus \Lambda_{N/8}$ and for *all* $0 \leqslant t \leqslant 1$), and in addition the last inequality holds due to (41). Thus, a simple union bound gives that the event $\{\sum_{v \in \mathcal{A}_{N/2}} (\langle \sigma_v^{\Lambda_N,+,1} \rangle_{\mu^{\Lambda_N,+,1}} - \langle \sigma_v^{\Lambda_N,-,0} \rangle_{\mu^{\Lambda_N,-,0}}) \leqslant r\theta^*\}$ contains an event $\mathcal{E}_{\mathcal{A}_{N/2}}$ which is measurable with respect to $\{h_v : v \notin \Lambda_{N/8}\}$ such that

$$\mathbb{P}(\mathcal{E}_{\mathcal{A}_{N/2}}) \geqslant 1/2. \tag{51}$$

Furthermore, let $\mathcal{T} = \{1 \leqslant t \leqslant 1 : m^{\Lambda_{N/8},+,t} - m^{\Lambda_{N/8},-,t} \geqslant \theta^*\}$. By (41) we have $\mathbb{E}|\mathcal{T}| \geqslant 1/2r$ where $|\mathcal{T}|$ is the Lebesgue measure of $\mathcal{T}$. Since $|\mathcal{T}| \leqslant 1$, we have $\mathbb{P}(|\mathcal{T}| \geqslant 1/4r) \geqslant 1/4r$. Therefore,

$$\mathbb{P}(\int_0^1 (m^{\Lambda_{N/8},+,t} - m^{\Lambda_{N/8},-,t}) dt \geqslant \theta^*/4r) \geqslant 1/4r. \tag{52}$$

Combined with (51), this yields that

$$\mathbb{P}(\mathcal{E}^\diamond) \geqslant 1/8r \tag{53}$$

where $\mathcal{E}^\diamond$ is the event such that

$$\int_0^1 (m^{\Lambda_{N/8},+,t} - m^{\Lambda_{N/8},-,t})dt$$
$$\geqslant \frac{\theta^*}{4r} \geqslant (4r^2)^{-1} \sum_{v\in\mathcal{A}_{N/2}} (\langle\sigma_v^{\Lambda_N,+,1}\rangle_{\mu^{\Lambda_N,+,1}} - \langle\sigma_v^{\Lambda_N,-,0}\rangle_{\mu^{\Lambda_N,-,0}}).$$

Suppose (42) does not hold. Then by (49) and the preceding display, the events $\mathcal{E}^\star$ and $\mathcal{E}^\diamond$ are mutually exclusive. But by (40) and (53), we have $\mathbb{P}(\mathcal{E}^\star)+\mathbb{P}(\mathcal{E}^\diamond) > 1$, arriving at a contradiction. □

Proof of Proposition 3.1 The Proof of Proposition 3.1 at this point is highly similar to that of Proposition 2.2. As a result, we only provide a sketch emphasizing the additional subtleties.

Let π be an arbitrary monotone coupling of $\mu^{\Lambda_N,\pm}$ and let $\mathcal{C}^{\Lambda_N} = \mathcal{C}^{\Lambda_N,\pi}$ be defined as in (25).

For any rectangle $A \subset \mathbb{R}^2$ (whose sides are not necessarily parallel to the axes), recall that ℓ_A is the length of the longer side and A^{Large} is the square box concentric with A and of side length $32\ell_A$. In addition, the aspect ratio of A is the ratio between the lengths of the longer and shorter sides. Consider an arbitrary rectangle A with aspect ratio at least $a = 100$. For a (random) set $\mathcal{C} \subset \mathbb{Z}^2$, we continue to use $\text{Cross}(A, \mathcal{C})$ to denote the event that there exists a path $v_0, \ldots, v_k \in A \cap \mathcal{C}$ connecting the two shorter sides of A. For *any* monotone coupling $\pi^{A^{\text{Large}}}$ of $\mu^{A^{\text{Large}},\pm}$ (below we denote $\mathcal{C}^{A^{\text{Large}}} = \{v \in A^{\text{Large}} : \sigma^{A^{\text{Large}},+} > \sigma^{A^{\text{Large}},-}\}$ under $\pi^{A^{\text{Large}}}$), we can adapt the Proof of (17) and deduce that (write $N' = \min\{2^n : 2^{n+2} \geqslant \ell_A\}$, and recall $\mathcal{E}^+$ as in Lemma 3.3)

$$\begin{aligned}
&\mathbb{P}\otimes\mu^{\Lambda_{N'},+}(\mathcal{E}^+_{N'})\\
&\quad\geqslant 1-4(1-\mathbb{P}\otimes\pi^{A^{\text{Large}}}(\text{Cross}(A,\mathcal{V}^{\sigma^{A^{\text{Large}},+},+})))\\
&\quad\geqslant 1-4(1-\mathbb{P}\otimes\pi^{A^{\text{Large}}}(\text{Cross}(A,\mathcal{C}^{A^{\text{Large}}}))),
\end{aligned}$$

where the second inequality follows from the fact that $\text{Cross}(A, \mathcal{C}^{A^{\text{Large}}}) \subset \text{Cross}(A, \mathcal{V}^{\sigma^{A^{\text{Large}},+},+})$. In addition, by a similar derivation of (45),

$$\begin{aligned}
&\mathbb{P}\otimes\pi^{A^{\text{Large}}}(\text{Cross}(A,\mathcal{C}^{A^{\text{Large}}}))\\
&\quad\leqslant \mathbb{P}\otimes\pi^{A^{\text{Large}}}(\#(\mathcal{C}^{A^{\text{Large}}}\cap A)\geqslant \ell_A/2)\\
&\quad\leqslant \frac{1}{2}(1+\mathbb{P}(\sum_{v\in\Lambda_{N'/8}}(\langle\sigma_v^{\Lambda_{N'},+}\rangle_{\mu^{\Lambda_{N'},+}} - \langle\sigma_v^{\Lambda_{N'},-}\rangle_{\mu^{\Lambda_{N'},-}}) > 10^{-3}N')).
\end{aligned}$$

Therefore, by Lemma 3.3,

$$\mathbb{P} \otimes \pi^{A^{\text{Large}}}(\text{Cross}(A, \mathcal{C}^{A^{\text{Large}}})) \leqslant 1 - \delta \quad \text{where } \delta = \delta(\varepsilon, \beta) > 0. \tag{54}$$

It is crucial that (54) holds uniformly for all possible monotone couplings $\pi^{A^{\text{Large}}}$. Note that the probability for $\text{Cross}(A, \mathcal{C}^{\Lambda_N,\pi})$ could potentially depend on the location of A, either due to different influences from the boundary at different locations or different coupling mechanisms chosen at different location. However, thanks to (54), all these probabilities have a uniform upper bound which is strictly less than 1. In addition, by monotonicity of the Ising model, for a collection of rectangles that are well-separated, the corresponding crossing events can be dominated by independent events which have probabilities strictly less than 1. Next, we complete the Proof of Proposition 3.1 by utilizing this intuition. For any $k \geqslant 1$ and any rectangles $A_1, \dots, A_k \subseteq \{v \in \mathbb{R}^2 : |v|_\infty \leqslant N/2\}$ with aspect ratios at least a such that (a) $\ell_0 \leqslant \ell_{A_i} \leqslant N/32$ for all $1 \leqslant i \leqslant k$ and (b) $A_1^{\text{Large}}, \dots, A_k^{\text{Large}}$ are disjoint, we see that under any coupling π of $\mu^{\Lambda_N,\pm}$, there exist sets $\mathcal{C}^{A_i^{\text{Large}}}$ such that

- $\mathcal{C}^{A_i^{\text{Large}}}$ is sampled according to *some* monotone coupling of $\mu^{A_i^{\text{Large}},\pm}$.
- $\mathcal{C}^{\Lambda_N,\pi} \cap A_i \subset \mathcal{C}^{A_i^{\text{Large}}} \cap A_i$ (by monotonicity of Ising model with respect to boundary conditions).
- $\mu^{A_i^{\text{Large}},\pm}$'s are mutually independent (as they only depend on $\{h_v : v \in A_i^{\text{Large}}\}$ respectively).

Therefore, by (54),

$$\mathbb{P} \otimes \pi(\cap_{i=1}^k \text{Cross}(A_i, \mathcal{C}^{A_i^{\text{Large}}})) \leqslant (1 - \delta)^k.$$

This proves an analogue of Lemma 2.4, which verifies the hypothesis required in order to apply [1]. The remaining proof is merely an adaption of Proposition 2.2 and thus we omit further details. □

3.2 Admissible coupling and adaptive admissible coupling

In Sects. 3.2 and 3.3, we wish to prove an analogue of Lemma 2.11. In the case for $T > 0$, it seems quite a bit more challenging as the choice of the coupling for various Ising measures plays a role, which seems to be subtle in light of Remark 3.8 below. To address the issue, we consider a general class of couplings for various Ising measures (i.e., adaptive admissible couplings) in this section. In Sect. 3.3.1, we describe a particular construction of adaptive admissible coupling, which is suited for the multi-scale analysis (the multi-scale analysis is a more complicated version of the Proof for Lemma 2.11) presented in Sect. 3.3.3.

For $k \geqslant 1$, we consider *deterministic* boundary conditions and external fields $(\tau^{(i)}, \{h_v^{(i)} : v \in \Lambda\})$ where $\tau^{(i)} \in \{-1, 1\}^{\partial\Lambda}$ for $1 \leqslant i \leqslant k$ (these will be fixed throughout this section). We define the partial order $\prec$ by

$$i \prec j \text{ if } \tau^{(i)} \leqslant \tau^{(j)} \quad \text{and} \quad h^{(i)} \leqslant h^{(j)}. \tag{55}$$

We say that $(\sigma^{(1)}, \dots, \sigma^{(k)})$ (for $\sigma^{(1)}, \dots, \sigma^{(k)} \in \{-1, 1\}^{\Lambda}$) is an admissible configuration if $\sigma^{(i)} \leqslant \sigma^{(j)}$ for all $i \prec j$. Denote by Σ_k the collection of all admissible configurations. For $A \subset \Lambda$, write $(\sigma^{(1)}, \dots, \sigma^{(k)})_A$ for the restriction of $(\sigma^{(1)}, \dots, \sigma^{(k)})$ on A.

Definition 3.7 For each $1 \leqslant i \leqslant k$, let $\mu^{(i)}$ be the Ising measure on Λ with boundary condition $\tau^{(i)}$ and external field $h^{(i)}$. We say that a measure π is an admissible coupling of $\mu^{(1)}, \dots, \mu^{(k)}$ if π is supported on Σ_k and its marginal distributions agree with $\mu^{(i)}$'s.

Remark 3.8 Ideally, it would be great if there would exist an admissible coupling π which satisfies the Markov field property. Or, it would also be great if there would exist an admissible coupling π which satisfies a weak version of Markov field property, such that for any $\Gamma \subset \Lambda$ the measure $\pi(\sigma_{S_\Gamma}^{(i)} \in \cdot \mid (\sigma^{(1)}, \dots, \sigma^{(k)})_\Gamma)$ is the Ising measure on S_Γ with boundary condition $\sigma_{\partial S_\Gamma}^{(i)}$ and external field $\{h_v^{(i)} : v \in S_\Gamma\}$. However, such coupling does not exist as we can see from the following simple example. Let us consider Ising measures on a line segment with no external field and plus/minus boundary conditions on one end (denoted as u). Suppose that there exists an admissible coupling π (in this case a monotone coupling) with weak Markov field property. Then conditioned on the event that the two spins disagree at the other end of the line (denoted as v), we claim that the spins from the two Ising measures have to disagree on every vertex on the line, thereby violating the weak Markov property. In order to verify the claim, we suppose the claim fails and let w be the first vertex (from u) where the two spins agree with each other. Conditioned on spins from u to w, the two marginals at v are the same (by the weak Markov property) and thus have to agree in a monotone coupling.

In light of Remark 3.8, we will seek for admissible couplings with a desirable property even weaker than the weak Markov field property. To this end, we will explore the spins using certain "adaptive" algorithm and then we will argue that the marginal measures on the unexplored region remain to be Ising measures. This motivates us to consider the *adaptive admissible coupling* (see Definition 3.9 below). Let $\Xi_k = \{(\sigma^{(1)}, \dots, \sigma^{(k)}) \in \{-1, 1\}^k : \sigma^{(i)} \leqslant$

$\sigma^{(j)}$ for all $i \prec j\}$. For $\theta_1, \ldots, \theta_k$ which are measures on $\{-1, 1\}$, we say that $\theta_1, \ldots, \theta_k$ are admissible if $\theta_i(1) \leqslant \theta_j(1)$ for all $i \prec j$. In this case, let θ be the monotone coupling of $\theta_1, \ldots, \theta_k$. That is, θ is the joint measure of $(\sigma_1, \ldots, \sigma_k)$, which is defined in terms of a uniform variable U on $[0, 1]$ such that

$$\sigma_i = -1 \quad \text{if and only if} \quad U \leqslant 1 - \theta_i(1).$$

Clearly, θ is supported on Ξ_k and its marginals are $\theta_1, \ldots, \theta_k$. In addition, θ is consistent, i.e.,

$$\begin{aligned} &\text{The projection of } \theta \text{ onto the first } (k-1) \\ &\quad \text{spins is the monotone coupling for } \theta_1, \ldots, \theta_{k-1}. \end{aligned} \tag{56}$$

In order to define adaptive admissible couplings, we make use of exploration procedures. An exploration procedure can be encoded by a family of deterministic maps $\{f_V : V \subset \Lambda, V \neq \Lambda\}$ where f_V is a mapping that maps an admissible configuration on V to a vertex in $\Lambda \setminus V$. That is to say, if we have explored a set $V \subset \Lambda$ and the spin configuration on V is given by $(\sigma^{(1)}, \ldots, \sigma^{(k)})_V$, then the next vertex we will explore is $f_V((\sigma^{(1)}, \ldots, \sigma^{(k)})_V)$.

Definition 3.9 For each exploration procedure $\{f_V\}$, we associate an admissible coupling in the following manner. Let $\mathcal{V}_0 = \emptyset$. For $t \geqslant 1$, let $v_t = f_{\mathcal{V}_{t-1}}((\sigma^{(1)}, \ldots, \sigma^{(k)})_{\mathcal{V}_{t-1}})$. Let $\mathcal{V}_t = \mathcal{V}_{t-1} \cup \{v_t\}$. Quenched on the realization of $\{\mathcal{V}_{t-1}, (\sigma^{(1)}, \ldots, \sigma^{(k)})_{\mathcal{V}_{t-1}}\}$, for $1 \leqslant i \leqslant k$ let $\theta_i^{(t)}(\pm 1) = \mu^{(i)}(\sigma_{v_t}^{(i)} = \pm 1 \mid \sigma_{\mathcal{V}_{t-1}}^{(i)})$. Let $\theta^{(t)}$ be the monotone coupling of $\theta_1^{(t)}, \ldots, \theta_k^{(t)}$, and we sample $(\sigma^{(1)}, \ldots, \sigma^{(k)})_{v_t}$ according to $\theta^{(t)}$. We repeat this procedure until $t = \#\Lambda$. We let π be the measure on $(\sigma^{(1)}, \ldots, \sigma^{(k)})$ at the end of the procedure. In addition, we say that a random set $\mathcal{V}$ is a *stopping set* if $\{\mathcal{V} = \mathcal{V}_t = V_t\}$ (for any deterministic $V_t \subset \Lambda$) is measurable with respect to $\{(\sigma^{(1)}, \ldots, \sigma^{(k)})_{V_t})\}$.

Remark 3.10 In the study of spin models, it is common to use an exploration procedure to discover certain observables (such as interfaces) associated with spin configurations. Often times, an instance of spin configurations is sampled a priori (which is usually sampled according to a Gibbs measure) and then the exploration procedure is performed on this instance. That being said, it is not uncommon to construct a measure as the exploration process evolves. Definition 3.9 is one example of such constructions, where the spin configuration is sampled as the exploration procedure evolves and more importantly the measure on spin configurations depends on the exploration procedure.

Lemma 3.11 *For each exploration procedure, the measure π given in Definition 3.9 is a well-defined admissible coupling. In addition, for any stopping set $\mathcal{V}$, given the realization of $\mathcal{V}$ and $(\sigma^{(1)}, \ldots, \sigma^{(k)})_{\mathcal{V}}$, the conditional measure of π restricted on $\mathcal{V}^c$ has marginals corresponding to Ising measures on $\mathcal{V}^c$ with boundary condition $\sigma^{(i)}_{\partial \mathcal{V}^c}$ and external field $\{h^{(i)}_v : v \in \mathcal{V}^c\}$.*

Proof The measure π is well-defined since we can inductively verify that for $t = 0, 1, 2, \ldots$, the sequence $\theta^{(t)}_1, \ldots, \theta^{(t)}_k$ is admissible and thus $(\sigma^{(1)}, \ldots, \sigma^{(k)})_{\mathcal{V}_{t+1}}$ is admissible. To prove the second part of the statement, it suffices to show that for each $1 \leqslant i \leqslant k$ and $1 \leqslant t \leqslant \#\Lambda$,

$$\begin{aligned} &\pi(\sigma^{(i)}_{\Lambda \setminus V_{t-1}} \in \cdot \mid (\sigma^{(1)}, \ldots, \sigma^{(k)})_{\mathcal{V}_{t-1}}, \mathcal{V}_{t-1} = V_{t-1}) \\ &\quad = \mu^{(i)}(\sigma^{(i)}_{\Lambda \setminus V_{t-1}} \in \cdot \mid \sigma^{(i)}_{V_{t-1}}). \end{aligned} \tag{57}$$

We prove (57) by induction for $t = \#\Lambda, \ldots, 1$. It is obvious from Definition 3.9 that (57) holds for $t = \#\Lambda$. Suppose (57) holds for t, we then deduce for $t-1$ that

$$\begin{aligned} &\pi(\sigma^{(i)}_{\Lambda \setminus (V_{t-2} \cup \{v_{t-1}\})} \in \cdot, \sigma^{(i)}_{v_{t-1}} = \pm 1 \mid (\sigma^{(1)}, \ldots, \sigma^{(k)})_{\mathcal{V}_{t-2}}, \mathcal{V}_{t-2} = V_{t-2}) \\ &= \mu^{(i)}(\sigma^{(i)}_{v_{t-1}} = \pm 1 \mid \sigma^{(i)}_{V_{t-2}}) \times \mu^{(i)}(\sigma^{(i)}_{\Lambda \setminus (V_{t-2} \cup \{v_{t-1}\})} \in \cdot \mid \sigma^{(i)}_{V_{t-2}}, \sigma^{(i)}_{v_{t-1}} = \pm 1). \end{aligned}$$

This implies that $\pi(\sigma^{(i)}_{\Lambda \setminus V_{t-2}} \in \cdot \mid (\sigma^{(1)}, \ldots, \sigma^{(k)})_{\mathcal{V}_{t-2}}, \mathcal{V}_{t-2} = V_{t-2}) = \mu^{(i)}(\sigma^{(i)}_{\Lambda \setminus V_{t-2}} \in \cdot \mid \sigma^{(i)}_{V_{t-2}})$, thereby completing the proof by induction. □

In what follows, we refer to π as in Definition 3.9 as an adaptive admissible coupling. In addition, we will always define adaptive admissible couplings by presenting an exploration procedure and then consider the associated admissible coupling given in Definition 3.9. For convenience of exposition, we usually describe an exploration procedure in words rather than specifying the maps $\{f_V\}$.

3.3 A multi-scale analysis via another perturbation argument

Let $\alpha > 1$ be as in Proposition 3.1. Let $\sqrt{1/\alpha} < \alpha' < 1$. Let $N_0 = N_0(\varepsilon, \beta)$ be a large number to be chosen. For each $N \geqslant N_0$ (of the form 4^n), set $\Delta = \Delta(N) = N^{-\alpha(\alpha')^2}$. In the rest of the paper, we consider the following perturbation:

$$\tilde{h}^{(N)}_v = \begin{cases} h_v + \Delta & \text{for } v \in \Lambda_N \setminus \Lambda_{N/4}, \\ h_v & \text{for } v \in \Lambda_{N/4}. \end{cases} \tag{58}$$

We denote by $\tilde{\mu}^{\Lambda_N,\pm}$ the Ising measures on Λ_N with respect to plus/minus boundary conditions and external field $\{\tilde{h}_v^{(N)} : v \in \Lambda_N\}$, and denote by $\tilde{\sigma}^{\Lambda_N,\pm}$ the spins sampled according to $\tilde{\mu}^{\Lambda_N,\pm}$. In this whole section except in (69) and (70), we will quench on the realization of $\{h_v\}$ and thus the external field is viewed as deterministic.

3.3.1 A construction of an adaptive admissible coupling

We will define the following adaptive admissible coupling π_{Λ_N} for $\mu^{\Lambda_N,\pm}$ and $\tilde{\mu}^{\Lambda_N,\pm}$. According to Definition 3.9, in order to specify π_{Λ_N}, we only need to specify the exploration procedure (i.e., the order of vertices in which we sample the spins), as described as follows. Throughout the procedure, we let $\mathcal{C}_*^{\Lambda_N}$ be the collection of vertices v which have been sampled such that $\sigma_v^{\Lambda_N,+} > \sigma_v^{\Lambda_N,-}$ and $\tilde{\sigma}_v^{\Lambda_N,+} > \tilde{\sigma}_v^{\Lambda_N,-}$. We first sample spins at vertices on $\partial\Lambda_k$ for $k = N-1, N-2, \ldots, \frac{N}{2}$. For vertices on $\partial\Lambda_k$, for concreteness we sample in clockwise order starting from the right top corner. Next, let $K = \lfloor N^{\alpha'\alpha} \rfloor$ and $\ell = \lfloor \frac{1}{4} N^{1-\alpha'} \rfloor$. A comment on the order of the scales chosen: the exploration procedure below contains ℓ phases, and in every phase we consider an annulus where the inner and outer boundaries have Euclidean distance $N^{\alpha'}$ and thus by Proposition 3.1 typically have intrinsic distance $\geqslant K \gg N$. This is why we can hope to gain a contraction when comparing the number of disagreements on an annulus to that on its neighboring (larger) annulus (see (71) below).

We now turn to the description of the exploration procedure. For each $1 \leqslant j \leqslant \ell$ our construction employs the following procedure which we refer to as Phase j (see Fig. 4 for an illustration). Let $N' = \frac{N}{2} - (j-1)N^{\alpha'}$.

- We set $A_{j,0} = \partial\Lambda_{N'} \cap \mathcal{C}_*^{\Lambda_N}$, $V_{j,0} = \Lambda_N \setminus \Lambda_{N'}$, and for $k = 0, 1, \ldots, K$, we inductively employ the following procedure (which we refer to as stage). At the beginning of Stage $k+1$, we first set $A_{j,k+1} = \emptyset$ and $V_{j,k+1} = V_{j,k}$.
  - If $A_{j,k} = \emptyset$ (which we denote as event $\mathcal{E}_{j,k,\emptyset}$), we sample the unexplored vertices in Λ_N in a prefixed order (which can be arbitrary) and stop our procedure. Otherwise, we explore all the neighbors of $A_{j,k}$ (in a certain prefixed order, which can be arbitrary) which are in $\Lambda_{N'} \setminus V_{j,k}$ (that is, vertices which have not been explored) and sample the spins at these vertices. We also put these vertices into $V_{j,k+1}$.
  - If a newly sampled vertex is in $\partial\Lambda_{N'-N^{\alpha'}}$ (we denote this as event $\mathcal{E}_{j,k,d}$, where the subscript d suggests an event related to the intrinsic distance), we sample the unexplored vertices in Λ_N in a prefixed order (which can be arbitrary) and stop our procedure. Otherwise, if a newly sampled vertex ends up in $\mathcal{C}_*^{\Lambda_N}$ then we add it to $A_{j,k+1}$. (For $k \geqslant 1$, it is

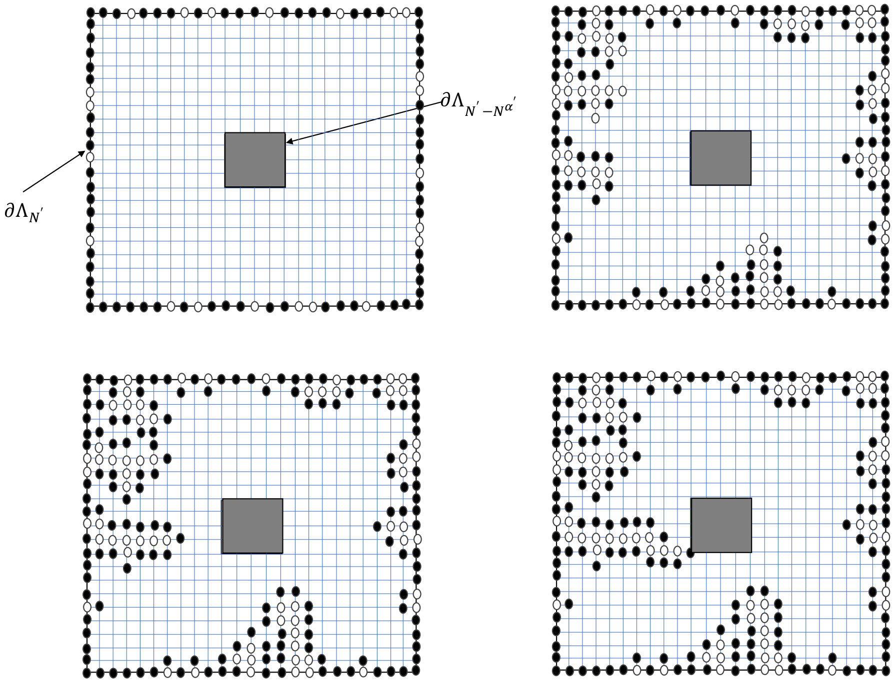

Fig. 4 Illustration for Phase j of the construction in Sect. 3.3.1. The inside square is $\Lambda_{N'-N^{\alpha'}}$, whose size has been reduced in the picture for better demonstration. On lattice points, empty indicates an unexplored vertex, an open circle indicates a vertex in $\mathcal{C}_*^{\Lambda_N}$, and a solid disk indicates a vertex not in $\mathcal{C}_*^{\Lambda_N}$. The top-left illustrates the beginning of Phase j, where vertices on $\partial\Lambda_{N'}$ have been explored (vertices outside have been explored too but we did not draw); the top-right illustrates the middle of Phase j (here $k=5$); the bottom-left picture illustrates event $\mathcal{E}_{j,k,\emptyset}$ (here $k=8$); the bottom right event illustrates $\mathcal{E}_{j,k,d}$ (here $k=12$)

clear that $A_{j,k}$ records all the vertices in $\Lambda_{N'}$ that are of $d_{\mathcal{C}_*^{\Lambda_N}}$-distance k to $\partial\Lambda_{N'}$ and $V_{j,k}$ records all the explored vertices up to Stage k.)

- Sample the unexplored vertices in $\Lambda_{N'} \setminus \Lambda_{N'-N^{\alpha'}}$ in a prefixed order (which can be arbitrary).

Finally, if the procedure is not yet stopped after ℓ phases, we sample the unexplored vertices in Λ_N in a prefixed order (which can be arbitrary).

Remark 3.12 (1) Later in the analysis, when we refer to sets such as $A_{j,k}$, $V_{j,k}$ we mean to use their values at the end of our procedure. (2) Note that in the preceding procedure, unless some event of the form $\mathcal{E}_{j,k,\emptyset}$ or $\mathcal{E}_{j,k,d}$ occurred, the exploration in all the ℓ phases is within $\Lambda_N \setminus \Lambda_{N/4}$.

3.3.2 Another perturbation argument

We use $\tilde{H}^{\Lambda_N,\pm}$, $\tilde{F}^{\Lambda_N,\pm}$, $\tilde{\sigma}^{\Lambda_N,\pm}$ to denote tilde versions of $H^{\Lambda_N,\pm}$, $F^{\Lambda_N,\pm}$, $\sigma^{\Lambda_N,\pm}$, i.e., defined analogously but with respect to the field $\{\tilde{h}_v^{(N)}\}$ defined as in (58). Without further notice, we will always consider measures where we couple all these Ising spins together. Thus, in particular, $\mathcal{C}^{\Lambda_N}$ and $\tilde{\mathcal{C}}^{\Lambda_N}$ are defined in the same probability space and we can then define $\mathcal{C}_*^{\Lambda_N} = \tilde{\mathcal{C}}^{\Lambda_N} \cap \mathcal{C}^{\Lambda_N}$.

We need some preparation before presenting our perturbative analysis. Suppose that $\mathcal{V}$ is a stopping set (see Definition 3.9) obtained when constructing π_{Λ_N} described in Sect. 3.3.1. Let $\pi'_{\mathcal{V}^c}$ be the restriction of π_{Λ_N} to $\mathcal{V}^c$. (We use prime in the notation $\pi'_{\mathcal{V}^c}$ as we wish to save $\pi_{\mathcal{V}^c}$ for later use.) By Lemma 3.11 and our definition of π_{Λ_N}, we see that $\pi'_{\mathcal{V}^c}$ depends on $(\sigma^{\Lambda_N,\pm})_{\mathcal{V}}$, $(\tilde{\sigma}^{\Lambda_N,\pm})_{\mathcal{V}}$ only through $(\sigma^{\Lambda_N,\pm})_{\partial\mathcal{V}^c}$, $(\tilde{\sigma}^{\Lambda_N,\pm})_{\partial\mathcal{V}^c}$. Thus, we may denote by $(\sigma^{\mathcal{V}^c,(\sigma^{\Lambda_N,\pm})_{\partial\mathcal{V}^c}}, \tilde{\sigma}^{\mathcal{V}^c,(\tilde{\sigma}^{\Lambda_N,\pm})_{\partial\mathcal{V}^c}})$ the spin configurations sampled according to $\pi'_{\mathcal{V}^c}$ with corresponding boundary conditions on $\partial\mathcal{V}^c$. Thus,

$$((\sigma_{\mathcal{V}}^{\Lambda_N,\pm}, \sigma^{\mathcal{V}^c,(\sigma^{\Lambda_N,\pm})_{\partial\mathcal{V}^c}}), (\tilde{\sigma}_{\mathcal{V}}^{\Lambda_N,\pm}, \tilde{\sigma}^{\mathcal{V}^c,(\tilde{\sigma}^{\Lambda_N,\pm})_{\partial\mathcal{V}^c}})) \text{ has law } \pi_{\Lambda_N}. \tag{59}$$

In what follows, we will mainly consider the measure $\pi'_{\mathcal{V}^c}$. For clarity of exposition, we quench on the realization of $\mathcal{V} = V$. Let $S = V^c$ and $\Gamma = \partial S$ (thus we have $S \subset S_\Gamma$). Further, we quench on the values of $(\sigma^{\Lambda_N,\pm})_\Gamma$, $(\tilde{\sigma}^{\Lambda_N,\pm})_\Gamma$ by

$$(\sigma^{\Lambda_N,\pm})_\Gamma = \tau^\pm, (\tilde{\sigma}^{\Lambda_N,\pm})_\Gamma = \tilde{\tau}^\pm, \quad \text{where } \tau^\pm, \tilde{\tau}^\pm \in \{-1, 1\}^\Gamma. \tag{60}$$

For $v \in \Gamma$ (in fact, any $v \in \Lambda_N$), by admissibility there are only six possible values for $(\tau_v^+, \tau_v^-, \tilde{\tau}_v^+, \tilde{\tau}_v^-)$ as shown in Table 1. For each such possible spin value, we will define a "hat" version $(\hat{\tau}_v^+, \hat{\tau}_v^-, \hat{\tilde{\tau}}_v^+, \hat{\tilde{\tau}}_v^-)$, where the definition is given in Table 2. Note that the hat version is a modification of the original spin value, and we emphasize the change in Table 2 by circling out the modifications. We will explain why we introduced the hat version of the spin on Γ after a number of definitions. From Tables 1 and 2, we see that

$$\hat{\tau}^+ \geqslant \hat{\tau}^- \geqslant \tau^-, \tilde{\tau}^+ \geqslant \hat{\tilde{\tau}}^+ \geqslant \hat{\tilde{\tau}}^-, \qquad \hat{\tilde{\tau}}^+ = \hat{\tau}^+ \geqslant \tau^+, \hat{\tilde{\tau}}^- = \hat{\tau}^- = \tilde{\tau}^-. \tag{61}$$

From a notation point of view, despite the fact that $\hat{\tau}^\pm = \hat{\tilde{\tau}}^\pm$, we still differentiate these two notations because our mental picture is that the boundary conditions $\hat{\tau}^\pm$ are matched to external field $\{h_v\}$ and the boundary conditions $\hat{\tilde{\tau}}^\pm$ are matched to external field $\{\tilde{h}_v^{(N)}\}$.

Recall that π'_S is the admissible coupling for Ising measures with boundary conditions and external fields $((\tau^\pm)_\Gamma, \{h_v\})$, $((\tilde{\tau}^\pm)_\Gamma, \{\tilde{h}_v^{(N)}\})$, where the order

Table 1 Original spins on Γ

type	τ_v^+	τ_v^-	$\tilde{\tau}_v^+$	$\tilde{\tau}_v^-$
a.	-1	-1	-1	-1
b.	-1	-1	$+1$	-1
c.	-1	-1	$+1$	$+1$
d.	$+1$	$+1$	$+1$	$+1$
e.	$+1$	-1	$+1$	$+1$
f.	$+1$	-1	$+1$	-1

Table 2 The hat version of the spins on Γ

type	$\hat{\tau}_v^+$	$\hat{\tau}_v^-$	$\hat{\tilde{\tau}}_v^+$	$\hat{\tilde{\tau}}_v^-$
a.	-1	-1	-1	-1
$\bullet b.$	-1	-1	-1	-1
$\bullet c.$	+1	+1	$+1$	$+1$
d.	$+1$	$+1$	$+1$	$+1$
$\bullet e.$	$+1$	+1	$+1$	$+1$
f.	$+1$	-1	$+1$	-1

of sampling vertex is given by that of π_{Λ_N} conditioned on spin configurations on the stopping set $\mathcal{V} = V$. In addition, we can extend π'_S to an adaptive admissible coupling π_S for Ising measures with boundary conditions and external fields $((\tau^\pm)_\Gamma, \{h_v\}), ((\tilde{\tau}^\pm)_\Gamma, \{\tilde{h}_v^{(N)}\}), ((\hat{\tau}^\pm)_\Gamma, \{h_v\}), ((\hat{\tilde{\tau}}^\pm)_\Gamma, \{\tilde{h}_v^{(N)}\})$, where the order of sampling vertices is determined by the coupling π'_S. Let $(\sigma^{S,\tau^\pm}, \tilde{\sigma}^{S,\tilde{\tau}^\pm}, \sigma^{S,\hat{\tau}^\pm}, \tilde{\sigma}^{S,\hat{\tilde{\tau}}^\pm})$ be the spin configuration sampled according to π_S (note that we use the tilde symbol on σ to emphasize the dependence on the external field $\{\tilde{h}_v^{(N)}\}$; similarly for H and F below). By (56), we see that the projection of π_S onto $(\sigma^{S,\tau^\pm}, \tilde{\sigma}^{S,\tilde{\tau}^\pm})$ has measure π'_S. As a result, we will simply use π_S in what follows. We also let $H^{S,\tau^\pm}, \tilde{H}^{S,\tilde{\tau}^\pm}, H^{S,\hat{\tau}^\pm}, \tilde{H}^{S,\hat{\tilde{\tau}}^\pm}$ denote Hamiltonians for corresponding Ising spins. Similarly, we denote by $F^{S,\tau^\pm}, \tilde{F}^{S,\tilde{\tau}^\pm}, F^{S,\hat{\tau}^\pm}, \tilde{F}^{S,\hat{\tilde{\tau}}^\pm}$ the log-partition-functions of corresponding Ising measures. Define

$$\mathcal{C}^{S,\tau^\pm} = \{v \in S : \sigma_v^{S,\tau^+} = 1, \sigma_v^{S,\tau^-} = -1\}$$

and similarly define $\tilde{\mathcal{C}}^{S,\tilde{\tau}^\pm}, \mathcal{C}^{S,\hat{\tau}^\pm}, \tilde{\mathcal{C}}^{S,\hat{\tilde{\tau}}^\pm}$. Define $\mathcal{C}_*^{S,\tau^\pm,\tilde{\tau}^\pm} = \mathcal{C}^{S,\tau^\pm} \cap \tilde{\mathcal{C}}^{S,\tilde{\tau}^\pm}$ and $\mathcal{C}_*^{S,\hat{\tau}^\pm,\hat{\tilde{\tau}}^\pm} = \mathcal{C}^{S,\hat{\tau}^\pm} \cap \tilde{\mathcal{C}}^{S,\hat{\tilde{\tau}}^\pm}$.

Now we have necessary notations to explain the reason for introducing the hat version of the spins on Γ. We wish to bound $\#(\mathcal{C}_*^{\Lambda_N} \cap S \cap (\Lambda_N \setminus \Lambda_{N/4}))$ in terms of $\#(\mathcal{C}_*^{\Lambda_N} \cap \Gamma)$. One way to achieve this is to track the increment for the difference between the log-partition-functions with plus and minus boundary conditions when the external field is perturbed. We see that on the one hand, the increment for the difference between log-partition-functions can be bounded from below in terms of $\#(\mathcal{C}_*^{\Lambda_N} \cap S \cap (\Lambda_N \setminus \Lambda_{N/4}))$ (see Lemma 3.15); and on the other hand such increment can be bounded from above by the number of disagreements for spins on Γ with respect to the plus and minus boundary conditions. However, when approaching the upper bound, the spin values of Type b, c, e as in Table 1 will also contribute to the upper bound despite the fact that they do not belong to $\mathcal{C}_*^{\Lambda_N} \cap \Gamma$. To address this, we introduce the hat version of the spins, which are in agreement except on $\mathcal{C}_*^{\Lambda_N} \cap \Gamma$. A crucial feature as we will show in Lemma 3.13, is that under the admissible coupling π_S we have $\mathcal{C}_*^{S,\tau^\pm,\tilde{\tau}^\pm} \subset \mathcal{C}_*^{S,\hat{\tau}^\pm,\hat{\tilde{\tau}}^\pm}$. Therefore, the intended lower bound on the increment for the difference between log-partition-functions is still valid for the hat version. Another crucial feature of the hat version of the spin is that

$$\begin{aligned}
&\{v \in \Gamma : \tau_v^+ = \tilde{\tau}_v^+ = 1, \tau_v^- = \tilde{\tau}_v^- = -1\} \\
&\quad = \{v \in \Gamma : \hat{\tau}_v^+ = \hat{\tilde{\tau}}_v^+ = 1, \hat{\tau}_v^- = \hat{\tilde{\tau}}_v^- = -1\} \\
&\quad = \{v \in \Gamma : \hat{\tau}_v^+ = 1, \hat{\tau}_v^- = -1\} = \{v \in \Gamma : \hat{\tilde{\tau}}_v^+ = 1, \hat{\tilde{\tau}}_v^- = -1\}.
\end{aligned} \tag{62}$$

Lemma 3.13 *Under the admissible coupling π_S, we have $\mathcal{C}_*^{S,\tau^\pm,\tilde{\tau}^\pm} \subset \mathcal{C}_*^{S,\hat{\tau}^\pm,\hat{\tilde{\tau}}^\pm}$.*

Proof For $u \in \mathcal{C}_*^{S,\tau^\pm,\tilde{\tau}^\pm}$, we have $\sigma_u^{S,\tau^+} = \tilde{\sigma}_u^{S,\tilde{\tau}^+} = 1$ and $\sigma_u^{S,\tau^-} = \tilde{\sigma}_u^{S,\tilde{\tau}^-} = -1$. By (61) and the admissible coupling, we see that $\sigma_u^{S,\hat{\tau}^+} \geqslant \sigma_u^{S,\tau^+} = 1$; similarly, $\sigma_u^{S,\hat{\tau}^-} \leqslant \tilde{\sigma}_u^{S,\tilde{\tau}^-} = -1$. So $u \in \mathcal{C}^{S,\hat{\tau}^\pm}$. In addition, by (61) and the admissible coupling, we see that $\tilde{\sigma}_u^{S,\hat{\tilde{\tau}}^+} \geqslant \sigma_u^{S,\tau^+} = 1$; similarly, $\tilde{\sigma}_u^{S,\hat{\tilde{\tau}}^-} = \tilde{\sigma}_u^{S,\tilde{\tau}^-} = -1$. So $u \in \tilde{\mathcal{C}}^{S,\hat{\tilde{\tau}}^\pm}$. Thus, $u \in \mathcal{C}_*^{S,\hat{\tau}^\pm,\hat{\tilde{\tau}}^\pm}$ as required. □

Corollary 3.14 *Under the admissible coupling π_S, we have $o \notin \mathcal{C}_*^{S,\tau^\pm,\tilde{\tau}^\pm}$ provided that $\mathcal{C}_*^{\Lambda_N} \cap \Gamma = \emptyset$.*

Proof If $\mathcal{C}_*^{\Lambda_N} \cap \Gamma = \emptyset$, we have $\hat{\tau}^+ = \hat{\tau}^- = \hat{\tilde{\tau}}^+ = \hat{\tilde{\tau}}^-$, in which case we have $\mathcal{C}_*^{S,\hat{\tau}^\pm,\hat{\tilde{\tau}}^\pm} = \emptyset$ and in particular $o \notin \mathcal{C}_*^{S,\hat{\tau}^\pm,\hat{\tilde{\tau}}^\pm}$. Combined with Lemma 3.13, this completes the proof of the corollary. □

Lemma 3.15 *We have that*

$$
\begin{aligned}
&2\Delta\langle \#(\mathcal{C}_*^{S,\hat{\tau}^\pm,\hat{\tilde{\tau}}^\pm} \cap (\Lambda_N \setminus \Lambda_{N/4}))\rangle_{\pi_S} \\
&\quad \leqslant (\tilde{F}^{S,\hat{\tilde{\tau}}^+} - \tilde{F}^{S,\hat{\tilde{\tau}}^-}) - (F^{S,\hat{\tau}^+} - F^{S,\hat{\tau}^-}) && (63)\\
&\quad \leqslant 16\#\{v \in \Gamma : \hat{\tau}_v^+ = \hat{\tilde{\tau}}_v^+ = 1, \hat{\tau}_v^- = \hat{\tilde{\tau}}_v^- = -1\}. && (64)
\end{aligned}
$$

Proof The proof of the lemma shares some similarity to that of Lemma 3.5. However, we give a self-contained proof here in order for clarity of exposition.

We first prove (64). A straightforward computation gives that

$$
\begin{aligned}
\tilde{F}^{S,\hat{\tilde{\tau}}^+} - \tilde{F}^{S,\hat{\tilde{\tau}}^-} &= \frac{1}{\beta}\log\frac{\sum_\sigma e^{-\beta \tilde{H}^{S,\hat{\tilde{\tau}}^+}(\sigma)}}{\sum_\sigma e^{-\beta \tilde{H}^{S,\hat{\tilde{\tau}}^-}(\sigma)}}\\
&\leqslant \frac{1}{\beta}\log e^{8\beta\cdot\#\{v\in\Gamma:\hat{\tilde{\tau}}_v^+\neq\hat{\tilde{\tau}}_v^-\}} \leqslant 8\cdot\#\{v \in \Gamma : \hat{\tilde{\tau}}_v^+ \neq \hat{\tilde{\tau}}_v^-\}.
\end{aligned}
$$

Similarly, $F^{S,\hat{\tau}^+} - F^{S,\hat{\tau}^-} \geqslant -8\cdot\#\{v \in \Gamma : \hat{\tau}_v^+ \neq \hat{\tau}_v^-\}$. Combined with (62), this proves (64).

Now we turn to prove (63). We write

$$
\begin{aligned}
&(\tilde{F}^{S,\hat{\tilde{\tau}}^+} - \tilde{F}^{S,\hat{\tilde{\tau}}^-}) - (F^{S,\hat{\tau}^+} - F^{S,\hat{\tau}^-})\\
&\quad = (\tilde{F}^{S,\hat{\tilde{\tau}}^+} - F^{S,\hat{\tau}^+}) - (\tilde{F}^{S,\hat{\tilde{\tau}}^-} - F^{S,\hat{\tau}^-}). && (65)
\end{aligned}
$$

For $0 \leqslant t \leqslant 1$, define

$$
\tilde{h}_v^{(t)} = \begin{cases} h_v + t\Delta & \text{for } v \in \Lambda_N \setminus \Lambda_{N/4},\\ h_v & \text{for } v \in \Lambda_{N/4}. \end{cases} \tag{66}
$$

Let $F^{S,\hat{\tau}^+,t}$ be the log-partition-function on S with boundary condition $\hat{\tau}^+$ (note that $\hat{\tilde{\tau}}^+ = \hat{\tau}^+$ by (61)) and external field $\{\tilde{h}_v^{(t)}\}$. In particular, $F^{S,\hat{\tau}^+,0} = F^{S,\hat{\tau}^+}$ and $F^{S,\hat{\tau}^+,1} = \tilde{F}^{S,\hat{\tilde{\tau}}^+}$. Similar notations apply for $F^{S,\hat{\tau}^-,t}$. Thus, we get that

$$
\tilde{F}^{S,\hat{\tilde{\tau}}^\pm} - F^{S,\hat{\tau}^\pm} = \int_0^1 \frac{dF^{S,\hat{\tau}^\pm,t}}{dt}dt. \tag{67}
$$

Denote by $\sigma^{S,\hat{\tau}^\pm,t}$ spins sampled according to Ising measures with boundary conditions $\hat{\tau}^\pm$ and external field $\{\tilde{h}^{(t)}\}$. In addition, for any fixed t, we let $\pi_{S,t}$ be the admissible coupling extended from π_S by also incorporating the spins

$\sigma^{S,\hat{\tau}^{\pm},t}$ (again, the order of sampling vertex is given by that of π_S). Therefore, we see

$$\frac{dF^{S,\hat{\tau}^{\pm},t}}{dt} = \Delta \sum_{v \in S \cap (\Lambda_N \setminus \Lambda_{N/4})} \langle \sigma_v^{S,\hat{\tau}^{\pm},t} \rangle_{\pi_{S,t}}.$$

Combined with (67) and (65), it yields that

$$\begin{aligned}&(F^{S,\hat{\hat{\tau}}^{+}} - F^{S,\hat{\hat{\tau}}^{-}}) - (F^{S,\hat{\tau}^{+}} - F^{S,\hat{\tau}^{-}})\\ &\quad = 2\int_0^1 \Delta \langle \#\{v \in S \cap (\Lambda_N \setminus \Lambda_{N/4}) : \sigma_v^{S,\hat{\tau}^{+},t} \neq \sigma_v^{S,\hat{\tau}^{-},t}\}\rangle_{\pi_{S,t}} dt. \quad (68)\end{aligned}$$

For any $v \in S$ and $t \in (0,1)$, by admissible coupling we have $\sigma_v^{S,\hat{\tau}^{+}} \leqslant \sigma_v^{S,\hat{\tau}^{+},t} \leqslant \tilde{\sigma}_v^{S,\hat{\hat{\tau}}^{+}}$ and $\sigma_v^{S,\hat{\tau}^{-}} \leqslant \sigma_v^{S,\hat{\tau}^{-},t} \leqslant \tilde{\sigma}_v^{S,\hat{\hat{\tau}}^{-}}$. Therefore,

$$\{v \in S \cap (\Lambda_N \setminus \Lambda_{N/4}) : \sigma_v^{S,\hat{\tau}^{+},t} \neq \sigma_v^{S,\hat{\tau}^{-},t}\} \supset \mathcal{C}_*^{S,\hat{\tau}^{\pm},\hat{\hat{\tau}}^{\pm}} \cap (\Lambda_N \setminus \Lambda_{N/4}).$$

Combined with (68), this completes the Proof of (63). □

Corollary 3.16 *Conditioned on the realization of the stopping set $\mathcal{V} = V$, let $S = V^c$ and $\Gamma = \partial S$. Then we have*

$$\Delta \langle \#(\mathcal{C}_*^{\Lambda_N} \cap S \cap (\Lambda_N \setminus \Lambda_{N/4})) \mid (\sigma^{\Lambda_N,\pm}, \tilde{\sigma}^{\Lambda_N,\pm})_V \rangle_{\pi_{\Lambda_N}} \leqslant 8\#\{\Gamma \cap \mathcal{C}_*^{\Lambda_N}\}.$$

Proof Quench on the realization of $(\sigma^{\Lambda_N,\pm}, \tilde{\sigma}^{\Lambda_N,\pm})_\Gamma$ as in (60). By Lemmas 3.13 and 3.15,

$$\begin{aligned}&\Delta \langle \#(\mathcal{C}_*^{S,\tau^{\pm},\tilde{\tau}^{\pm}} \cap (\Lambda_N \setminus \Lambda_{N/4})) \rangle_{\pi_S}\\ &\quad \leqslant 8\#\{v \in \Gamma : \hat{\tau}_v^{+} = \hat{\hat{\tau}}_v^{+} = 1, \hat{\tau}_v^{-} = \hat{\hat{\tau}}_v^{-} = -1\}\\ &\quad = 8\#\{v \in \Gamma : \tau_v^{+} = \tilde{\tau}_v^{+} = 1, \tau_v^{-} = \tilde{\tau}_v^{-} = -1\},\end{aligned}$$

where the equality follows from (62). Combined with (59), this completes the proof of the corollary. □

3.3.3 Analysis of the adaptive admissible coupling

We now analyze the adaptive admissible coupling π_{Λ_N}. Recall that $\ell = \lfloor \frac{1}{4} N^{1-\alpha'} \rfloor$ and $K = \lfloor N^{\alpha\alpha'} \rfloor$, and define $\mathcal{D}_N$ to be the event (measurable with respect to the Gaussian field) by

$$\mathcal{D}_N = \{\pi_{\Lambda_N}(\min_{1\leqslant j\leqslant \ell} d_{\mathcal{C}^{\Lambda_N}}(\partial\Lambda_{N/2-jN^{\alpha'}}, \partial\Lambda_{N/2-(j-1)N^{\alpha'}}) \leqslant K) \geqslant N^{-20}\}. \tag{69}$$

By Proposition 3.1 and a simple Markov's inequality, we see that for $C = C(\varepsilon, \beta) > 0$

$$\mathbb{P}(\mathcal{D}_N) \leqslant CN^{-20}. \tag{70}$$

In what follows, we quench on the Gaussian field at which $\mathcal{D}_N$ does not occur.

Lemma 3.17 *We have that $\pi_{\Lambda_N}(o \in \mathcal{C}_*^{\Lambda_N}) \leqslant CN^{-10}$ on $\mathcal{D}_N^c$, for $C = C(\varepsilon, \beta) > 0$.*

Proof For $1 \leqslant j \leqslant \ell$, $1 \leqslant k \leqslant K$, let $\mathcal{E}_{j,k,\emptyset}$, $\mathcal{E}_{j,k,d}$, $V_{j,k}$, $A_{j,k}$ be defined as in Sect. 3.3.1. For each $1 \leqslant j \leqslant \ell$, let $\mathcal{E}_{j,\emptyset} = \cup_{i=1}^{j} \cup_{k=1}^{K} \mathcal{E}_{i,k,\emptyset}$ and define

$$m_j^* = \langle \#(\mathcal{C}_*^{\Lambda_N} \cap (\Lambda_{N/2-(j-1)N^{\alpha'}} \setminus \Lambda_{N/2-jN^{\alpha'}}))\mathbf{1}_{\mathcal{E}_{j-1,\emptyset}^c}\rangle_{\pi_{\Lambda_N}}.$$

By Corollary 3.14, it suffices to prove that $m_\ell^* \leqslant 2N^{-10}$. To this end, it suffices to prove that for $N \geqslant N_0 = N_0(\varepsilon, \beta)$ (where N_0 is to be selected)

$$m_{j+1}^* \leqslant 10^{-3} m_j^* + N^{-10} \quad \text{for all } 1 \leqslant j \leqslant \ell - 1. \tag{71}$$

Let $\mathcal{E}_{j,d} = \cup_{i=1}^{j} \cup_{k=1}^{K} \mathcal{E}_{i,k,d}$. Since $\pi_{\Lambda_N}(\mathcal{E}_{j,d}) \leqslant CN^{-20}$ on $\mathcal{D}_N^c$, it suffices to show that

$$\langle \#(\mathcal{C}_*^{\Lambda_N} \cap (\Lambda_{N/2-jN^{\alpha'}} \setminus \Lambda_{N/2-(j+1)N^{\alpha'}}))\mathbf{1}_{\mathcal{E}_{j,\emptyset}^c}\mathbf{1}_{\mathcal{E}_{j,d}^c}\rangle_{\pi_{\Lambda_N}} \leqslant 10^{-3} m_j^*. \tag{72}$$

Fix $1 \leqslant j \leqslant \ell$. For $1 \leqslant k \leqslant K$, write $\mathcal{E}_{j,\leqslant k,\emptyset} = \mathcal{E}_{j-1,\emptyset} \cup \cup_{i=1}^{k} \mathcal{E}_{j,i,\emptyset}$ and $\mathcal{E}_{j,\leqslant k,d} = \mathcal{E}_{j-1,d} \cup \cup_{i=1}^{k} \mathcal{E}_{j,i,d}$. Thus, we can deduce that

$$\begin{aligned}
&\Delta\langle \#(\mathcal{C}_*^{\Lambda_N} \cap (\Lambda_{N/2-jN^{\alpha'}} \setminus \Lambda_{N/2-(j+1)N^{\alpha'}}))\mathbf{1}_{\mathcal{E}_{j,\leqslant k,\emptyset}^c}\mathbf{1}_{\mathcal{E}_{j,\leqslant k,d}^c} \\
&\quad \mid (\sigma^{\Lambda_N,\pm}, \tilde{\sigma}^{\Lambda_N,\pm})_{V_{j,k}}\rangle_{\pi_{\Lambda_N}} \\
&\quad = \mathbf{1}_{\mathcal{E}_{j,\leqslant k,\emptyset}^c}\mathbf{1}_{\mathcal{E}_{j,\leqslant k,d}^c}\Delta\langle \#(\mathcal{C}_*^{\Lambda_N} \cap (\Lambda_{N/2-jN^{\alpha'}} \setminus \Lambda_{N/2-(j+1)N^{\alpha'}})) \\
&\qquad \mid (\sigma^{\Lambda_N,\pm}, \tilde{\sigma}^{\Lambda_N,\pm})_{V_{j,k}}\rangle_{\pi_{\Lambda_N}} \\
&\quad \leqslant 8\#A_{j,k} \cdot \mathbf{1}_{\mathcal{E}_{j,\leqslant k,\emptyset}^c}\mathbf{1}_{\mathcal{E}_{j,\leqslant k,d}^c},
\end{aligned}$$

where the equality holds since $\mathcal{E}_{j,\leqslant k,\emptyset}$ and $\mathcal{E}_{j,\leqslant k,d}$ are measurable with respect to $(\sigma^{\Lambda_N,\pm}, \tilde{\sigma}^{\Lambda_N,\pm})_{V_{j,k}}$, and the inequality is obtained by applying Corollary 3.16 with $V = V_{j,k}$ (note that $\Lambda_{N/2-jN^{\alpha'}} \cap V_{j,k} = \emptyset$ on the event

$\mathcal{E}^c_{j,\leqslant k,d}$). Averaging over the conditioning in the preceding display and recalling that $\mathcal{E}_{j-1,\emptyset} \subset \mathcal{E}_{j,\leqslant k,\emptyset} \subset \mathcal{E}_{j,\emptyset}$ and $\mathcal{E}_{j,\leqslant k,d} \subset \mathcal{E}_{j,d}$, we deduce that

$$\begin{aligned}&\Delta\langle \#(\mathcal{C}_*^{\Lambda_N} \cap (\Lambda_{N/2-jN^{\alpha'}} \setminus \Lambda_{N/2-(j+1)N^{\alpha'}}))\mathbf{1}_{\mathcal{E}^c_{j,\emptyset}}\mathbf{1}_{\mathcal{E}^c_{j,d}}\rangle_{\pi_{\Lambda_N}}\\ &\quad\leqslant \langle 8\#A_{j,k}\cdot \mathbf{1}_{\mathcal{E}^c_{j-1,\emptyset}}\mathbf{1}_{\mathcal{E}^c_{j,\leqslant k,d}}\rangle_{\pi_{\Lambda_N}}.\end{aligned}$$

Since $\sum_{k=1}^K \#A_{j,k}\cdot \mathbf{1}_{\mathcal{E}^c_{j,\leqslant k,d}} \leqslant \#(\mathcal{C}_*^{\Lambda_N} \cap (\Lambda_{N/2-(j-1)N^{\alpha'}} \setminus \Lambda_{N/2-jN^{\alpha'}}))$, summing the preceding display over $1 \leqslant k \leqslant K$ yields (72) (recall that $\Delta K = N^{-\alpha(\alpha')^2}\lfloor N^{\alpha\alpha'}\rfloor \geqslant 10^5$ if $N \geqslant N_0$ for large enough N_0). This completes the proof of the lemma. □

3.4 Proof of Theorem 1.1 for $T > 0$

We continue to consider $\tilde{h}^{(N)}$ defined as in (58), and let $\mu^{\Lambda_N,\pm}$, $\tilde{\mu}^{\Lambda_N,\pm}$, π_{Λ_N} be defined as in Sect. 3.3. For $\delta > 0$, let $Q_\delta \subset [-1, 1]$ be the collection of multiples of δ, and for $q \in Q_\delta$ define $\mathcal{E}^*_{o,N,q}$ to be an event measurable with respect to the Gaussian field by (the tilde symbol only applies on the minus version below)

$$\mathcal{E}^*_{o,N,q} = \{\langle \sigma_o^{\Lambda_N,+}\rangle_{\mu^{\Lambda_N,+}} \geqslant q+\delta,\ \langle \tilde{\sigma}_o^{\Lambda_N,-}\rangle_{\tilde{\mu}^{\Lambda_N,-}} \leqslant q-\delta\}. \tag{73}$$

By admissibility, on the event $\mathcal{E}^*_{o,N,q}$ we have $\pi_{\Lambda_N}(o \in \mathcal{C}_*^{\Lambda_N}) \geqslant \delta$. Combined with Lemma 3.17 and (70), it yields that

$$\mathbb{P}(\mathcal{E}^*_{o,N,q}) = O(N^{-10}/\delta). \tag{74}$$

(Throughout, $O(1)$ hides a constant that may depend on (ε, β).) Next, we define

$$\mathcal{E}_{o,N,q} = \{\langle \sigma_o^{\Lambda_N,+}\rangle_{\mu^{\Lambda_N,+}} \geqslant q+\delta,\ \langle \sigma_o^{\Lambda_N,-}\rangle_{\mu^{\Lambda_N,-}} \leqslant q-\delta\}. \tag{75}$$

By monotonicity, we thus have

$$\mathcal{E}_{o,N,q} \subset \mathcal{E}_{o,N',q} \quad \text{and} \quad \mathcal{E}^*_{o,N,q} \subset \mathcal{E}^*_{o,N',q} \text{ for all } N' \leqslant N. \tag{76}$$

Lemma 3.18 *Let $\delta = N^{-3}/3$. There exists $C = C(\varepsilon, \beta) > 0$ such that $\mathbb{P}(\mathcal{E}_{o,N,q}) \leqslant CN^{-6}$ for all $q \in Q_\delta$.*

Proof While the proof of the lemma is similar to that of Lemma 2.14, we nevertheless provide a self-contained proof for clarity of exposition.

For $A \subseteq \mathbb{Z}^2$, we set $h_A = \sum_{v\in A} h_v$. Without loss of generality, let us only consider $N = 4^n$ for some $n \geqslant 1$, and for $1 \leqslant \ell \leqslant n$, we define

$\{\tilde{h}_v^{(4^\ell)} : v \in \Lambda_{4^\ell}\}$ as in (58). Write $\mathfrak{A}_\ell = \Lambda_{4^\ell} \setminus \Lambda_{4^{\ell-1}}$. For $0.9n \leqslant \ell \leqslant n$, let $\mathcal{F}_\ell = \sigma(h_v : v \in \Lambda_{4^\ell})$ and write

$$h_v = (\#\mathfrak{A}_\ell)^{-1} h_{\mathfrak{A}_\ell} + g_v \quad \text{for } v \in \mathfrak{A}_\ell, \tag{77}$$

where $\{g_v : v \in \mathfrak{A}_\ell\}$ is a mean-zero Gaussian process independent of $h_{\mathfrak{A}_\ell}$ and $\{g_v : v \in \mathfrak{A}_\ell\}$ for $0.9n \leqslant \ell \leqslant n$ are mutually independent. Let $\mathcal{F}'_\ell$ be the σ-field which contains every event in $\mathcal{F}_\ell$ that is independent of $h_{\mathfrak{A}_\ell}$ (so in particular $\mathcal{F}_\ell \subset \mathcal{F}'_{\ell+1} \subset \mathcal{F}_{\ell+1}$). Write $\mathcal{E}_* = \cup_{0.9n \leqslant \ell \leqslant n} \mathcal{E}^*_{o,4^\ell,q}$. By monotonicity of $\langle \sigma_o^{\Lambda_N,+} \rangle_{\mu^{\Lambda_N,+}}$ and $\langle \sigma_o^{\Lambda_N,-} \rangle_{\mu^{\Lambda_N,-}}$ with respect to the external field, there exists an interval I_ℓ measurable with respect to $\mathcal{F}'_\ell$ such that conditioned on $\mathcal{F}'_\ell$ we have $\mathcal{E}_{o,4^\ell,q}$ occurs if and only if $h_{\mathfrak{A}_\ell} \in I_\ell$. Let I'_ℓ be the maximal sub-interval of I_ℓ which shares the upper endpoint and $|I'_\ell| \leqslant \frac{\#\mathfrak{A}_\ell}{4^{\alpha(\alpha')^2 \ell}}$ (here $|I'_\ell|$ denotes the length of the interval I'_ℓ). By definition in (73) and (58), we see from (77) that conditioned on $\mathcal{F}'_\ell$ we have that $\mathcal{E}_{o,4^\ell,q} \cap (\mathcal{E}^*_{o,4^\ell,q})^c$ occurs only if $h_{\mathfrak{A}_\ell} \in I'_\ell$. Thus,

$$\mathbb{P}(\mathcal{E}_{o,4^\ell,q} \cap (\mathcal{E}^*_{o,4^\ell,q})^c \mid \mathcal{F}'_\ell) \leqslant \mathbb{P}(h_{\mathfrak{A}_\ell} \in I'_\ell), \quad \text{for } 0.9n \leqslant \ell \leqslant n.$$

Combined with the fact that $\mathrm{Var}(h_{\mathfrak{A}_\ell}) = \varepsilon^2 \#\mathfrak{A}_\ell$, this gives that for $C = C(\varepsilon, \beta) > 0$ (whose value may be adjusted below)

$$\mathbb{P}(\mathcal{E}_{o,4^\ell,q} \cap (\mathcal{E}^*_{o,4^\ell,q})^c \mid \mathcal{F}'_\ell) \leqslant \frac{C}{4^{\ell(\alpha(\alpha')^2 - 1)}}.$$

By (76), we have $\mathcal{E}_{o,N,q} \cap \mathcal{E}_*^c = \cap_{\ell=0.9n}^n (\mathcal{E}_{o,4^\ell,t} \cap (\mathcal{E}^*_{o,4^\ell,q})^c)$. Since $(\mathcal{E}_{o,4^\ell,t} \cap (\mathcal{E}^*_{o,4^\ell,q})^c)$ is $\mathcal{F}_\ell$-measurable (and thus is $\mathcal{F}'_{\ell+1}$-measurable), we deduce that (recalling $\alpha(\alpha')^2 > 1$)

$$\mathbb{P}(\mathcal{E}_{o,N,q} \cap \mathcal{E}_*^c) \leqslant C N^{-6}.$$

By (74), we have $\mathbb{P}(\mathcal{E}_*) \leqslant C N^{-6}$. Combined with the preceding display, this completes the proof of the lemma. □

Define $\mathcal{E}_{o,N}$ to be an event measurable with respect to the Gaussian field by

$$\mathcal{E}_{o,N} = \{\langle \sigma_o^{\Lambda_N,+} \rangle_{\mu^{\Lambda_N,+}} - \langle \sigma_o^{\Lambda_N,-} \rangle_{\mu^{\Lambda_N,-}} \geqslant N^{-3}\}. \tag{78}$$

Since $\mathcal{E}_{o,N} \subset \cup_{q \in Q_\delta} \mathcal{E}_{o,N,q}$ with $\delta = N^{-3}/3$, we get from Lemma 3.18 that $\mathbb{P}(\mathcal{E}_{o,N}) = O(N^{-3})$. Thus,

$$\begin{aligned}
&\mathbb{E}(\langle\sigma_o^{\Lambda_N,+}\rangle_{\mu^{\Lambda_N,+}} - \langle\sigma_o^{\Lambda_N,-}\rangle_{\mu^{\Lambda_N,-}})\\
&\quad\leqslant 2\mathbb{P}(\mathcal{E}_{o,N}) + \mathbb{E}(\mathbf{1}_{\mathcal{E}^c_{o,N}}(\langle\sigma_o^{\Lambda_N,+}\rangle_{\mu^{\Lambda_N,+}} - \langle\sigma_o^{\Lambda_N,-}\rangle_{\mu^{\Lambda_N,-}}))\\
&\quad= O(N^{-3}).
\end{aligned} \tag{79}$$

Remark 3.19 In Lemma 3.18, we work with $\mathcal{E}_{o,N,q}$ other than $\mathcal{E}_{o,N}$, for the reason that we do not have the property that $\mathcal{E}_{o,N}$ occurs if and only if $h_{\mathfrak{A}_{\ell+1}}$ is in a certain interval (but the property holds for $\mathcal{E}_{o,N,q}$).

In order to prove Theorem 1.1, we will consider a monotone coupling of $\mu^{\Lambda_N,\pm}$ and consider $\mathcal{C}^{\Lambda_N} = \{v \in \Lambda_N : \sigma_v^{\Lambda_N,+} > \sigma_v^{\Lambda_N,-}\}$. We wish to have that $\{o \in \mathcal{C}^{\Lambda_N}\}$ occurs only if o is connected to $\partial\Lambda_N$ in $\mathcal{C}^{\Lambda_N}$. However, as we have seen in Remark 3.8, this property does not hold for all monotone couplings of $\mu^{\Lambda_N,\pm}$ (For instance if we build an adaptive admissible coupling by first sampling the spin at o and then the rest of the spins, then it is possible to get a configuration where the spin disagrees at o but there exists a contour surrounding o where all spins agree on this contour). In order to address this issue, we will construct an adaptive admissible coupling $\bar{\pi}_{\Lambda_N}$ such that this percolation property holds. Our construction is similar to that in Sect. 3.3.1 in a way that we explore $\mathcal{C}^{\Lambda_N}$ in a breadth first search order. But our construction now is much simpler as we no longer need to consider multiple phases.

By Definition 3.9, in order to define $\bar{\pi}_{\Lambda_N}$ we only need to specify the order of vertices in which we sample the spins, as described as follows. Throughout the procedure, we let $\mathcal{C}^{\Lambda_N}$ be the collection of vertices v which have been sampled and satisfy $\sigma_v^{\Lambda_N,+} > \sigma_v^{\Lambda_N,-}$. We set $A_0 = \partial\Lambda_N$ and for $k = 0, 1, 2, \ldots$, we inductively employ the following procedure (which we refer to as stage).

- At stage $k+1$, first set $A_{k+1} = \emptyset$. If $A_k = \emptyset$, we sample the unexplored vertices in Λ_N in an (arbitrary) prefixed order and stop our procedure. Otherwise, we explore all the unexplored neighbors of A_k (in a certain arbitrary prefixed order) and sample the spins at these vertices.
- For each newly sampled vertex, if it is in $\mathcal{C}^{\Lambda_N}$ then we add it to A_{k+1}.

Lemma 3.20 *Under the coupling $\bar{\pi}_{\Lambda_N}$, $o \in \mathcal{C}^{\Lambda_N}$ only if o is connected to $\partial\Lambda_N$ in $\mathcal{C}^{\Lambda_N}$.*

Proof Let k_* be the first k such that $A_k = \emptyset$. If o has been explored by the end of Stage $(k_* - 1)$, we see that o is connected to $\partial\Lambda_N$ in $\mathcal{C}^{\Lambda_N}$. Otherwise, denote V_{k_*} the collection of explored vertices at the end of Stage (k_*). If o was explored in Stage k_*, then $o \notin \mathcal{C}^{\Lambda_N}$ (since $A_{k_*} = \emptyset$). If o was not explored by the end of Stage k^*, we see that $\sigma^{\Lambda_N,+}$ and $\sigma^{\Lambda_N,-}$ agree on $\partial V^c_{k_*}$, and thus they will have to agree with each other on $V^c_{k_*}$ by Lemma 3.11 (this is because $\sigma_v^{\Lambda_N,+}$ and $\sigma^{\Lambda_N,-}$ have the same conditional marginal for all $v \in V^c_{k_*}$ and thus have to agree with each other in an admissible coupling). This in particular implies that $o \notin \mathcal{C}^{\Lambda_N}$, completing the proof of the lemma. $\square$

Proof of Theorem 1.1: $T > 0$ Consider the adaptive admissible coupling $\bar{\pi}_{\Lambda_N}$. We will use the fact that $\mathbb{P} \otimes \bar{\pi}_{\Lambda_N}(v \in \mathcal{C}^{\Lambda_N}) = \frac{1}{2}\mathbb{E}(\langle \sigma_v^{\Lambda_N,+} \rangle_{\mu^{\Lambda_N,+}} - \langle \sigma_v^{\Lambda_N,-} \rangle_{\mu^{\Lambda_N,-}})$ for all $v \in \Lambda_N$. Let $N_0 = N_0(\varepsilon, \beta)$ be chosen later. For any box B, recall that B^{large} is the box concentric with B of doubled side length. For $B \in \mathcal{B}(N, N_0)$, we say B is open if $\mathcal{C}^{\Lambda_N} \cap B \neq \emptyset$. In order to analyze this percolation process, we say a box B is exceptional if $\sum_{v \in B}(\langle \sigma_v^{B^{\text{large}},+} \rangle_{\mu^{B^{\text{large}},+}} - \langle \sigma_v^{B^{\text{large}},-} \rangle_{\mu^{B^{\text{large}},-}}) \geqslant N_0^{-1/2}$ (so exceptional is a property measurable with respect to $\{h_v : v \in B^{\text{large}}\}$). By (79) and monotonicity,

$$\begin{aligned}
&\mathbb{P}(B \text{ is exceptional}) \\
&\quad \leqslant N_0^{1/2} \sum_{v \in B} \mathbb{E}(\langle \sigma_v^{B^{\text{large}},+} \rangle_{\mu^{B^{\text{large}},+}} - \langle \sigma_v^{B^{\text{large}},-} \rangle_{\mu^{B^{\text{large}},-}}) = O(N_0^{-1/2}).
\end{aligned}$$

Recall Definition 2.9. We see that the exceptional boxes on $\mathcal{B}(N, N_0)$ form a percolation process which satisfies the $(N, N_0, 4, p)$-condition with $p = O(N_0^{-1/2})$. In addition, for any box B which is not exceptional, denoting by $\mathcal{F}_B$ the σ-field generated by spin configurations outside B^{large}, we see from monotonicity that

$$\begin{aligned}
&\bar{\pi}_{\Lambda_N}(B \text{ is open } \mid \mathcal{F}_B) \\
&\quad \leqslant \sum_{v \in B} (\langle \sigma_v^{B^{\text{large}},+} \rangle_{\mu^{B^{\text{large}},+}} - \langle \sigma_v^{B^{\text{large}},-} \rangle_{\mu^{B^{\text{large}},-}}) = O(N_0^{-1/2}).
\end{aligned}$$

Altogether, this implies that the collection of open boxes forms a percolation process which also satisfies the $(N, N_0, 4, p)$-condition with $p = O(N_0^{-1/2})$. By Lemma 3.20, in order for $o \in \mathcal{C}^{\Lambda_N}$, it is necessary that there exists an open lattice animal on $B \in \mathcal{B}(N, N_0)$ with size at least $\frac{N}{10N_0}$. Now, choosing N_0 sufficiently large (so that p is sufficiently small) and applying Lemma 2.10 yields that

$$\mathbb{P} \otimes \bar{\pi}_{\Lambda_N}(o \in \mathcal{C}^{\Lambda_N}) \leqslant c^{-1} e^{-cN} \quad \text{for } c = c(\varepsilon, \beta) > 0,$$

completing the proof of the theorem. $\square$

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附件6：荣誉数学拔尖人才培养成果主要完成人 Charles Newman 教授及林芳华教授的授课记录

Charles Newman 教授授课记录一栏

Faculty	Term	Subject	Class Title
Charles Newman	Fall 2013	MATH-SHU	Analysis I
Charles Newman	Fall 2013	MATH-SHU	Analysis I
Charles Newman	Spring 2014	MATH-SHU	Analysis II
Charles Newman	Fall 2014	MATH-SHU	Functions of a Complex Variable
Charles Newman	Fall 2015	MATH-SHU	Honors Calculus
Charles Newman	Fall 2015	MATH-SHU	Honors Calculus
Charles Newman	Spring 2016	MATH-SHU	Honors Analysis I
Charles Newman	Fall 2016	MATH-SHU	Honors Calculus
Charles Newman	Fall 2016	MATH-SHU	Honors Calculus
Charles Newman	Spring 2017	MATH-SHU	Honors Analysis I
Charles Newman	Fall 2017	MATH-SHU	Honors Calculus
Charles Newman	Spring 2018	MATH-SHU	Honors Analysis I

Instructor Schedule

ID 10879463 Charles Newman

Term 1138 Fall 2013

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
20497	MATH-SHU	201	001	LEC	Analysis I	11:30AM	12:45PM	TuTh	GEOG	G153
20498	MATH-SHU	201	002	RCT	Analysis I	11:30AM	12:45PM	F	GEOG	G153

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Instructor Schedule

ID 10879463 Charles Newman

Term 1144 Spring 2014

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
20295	MATH-SHU	202	001	LEC	Analysis II	1:00PM	2:15PM	MW	GEOG	G149

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Instructor Schedule

ID 10879463 Charles Newman
Term 1148 Fall 2014

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
19199	MATH-SHU	282	001	LEC	Functions of a Complex Variabl	9:45AM	11:00AM	MW	PDNG	304

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Instructor Schedule

ID 10879463 Charles Newman
Term 1158 Fall 2015

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
17133	MATH-GA	2931	002	LEC	Adv Tpcs in Probability:	1:25PM	3:15PM	WF	Cour Inst/	202
13318	MATH-SHU	201	001	LEC	Honors Calculus	2:45PM	4:00PM	MW	PDNG	309
13319	MATH-SHU	201	002	RCT	Honors Calculus	2:45PM	4:00PM	F	PDNG	309

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Instructor Schedule

ID 10879463 Charles Newman

Term 1164 Spring 2016

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
14595	MATH-SHU	328	001	LEC	Honors Analysis I	11:15AM	12:30PM	MW	PDNG	207

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Instructor Schedule

ID 10879463 Charles Newman

Term 1168 Fall 2016

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
2046	MATH-GA	2931	002	LEC	Adv Tpcs in Probability:	1:25PM	3:15PM	WF	Cour Inst/	202
12973	MATH-SHU	201	001	LEC	Honors Calculus	9:00AM	12:00PM	Th		
12973	MATH-SHU	201	001	LEC	Honors Calculus	2:45PM	4:00PM	MW	PDNG	209

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Instructor Schedule

ID 10879463 Charles Newman

Term 1174 Spring 2017

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
13909	MATH-SHU	328	001	LEC	Honors Analysis I	11:15AM	12:30PM	MW	PDNG	212

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Instructor Schedule

ID 10879463 Charles Newman

Term 1178 Fall 2017

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
1440	MATH-GA	2931	001	LEC	Adv Tpcs in Probability:	1:25PM	3:15PM	MW	Cour Inst/	705
19030	MATH-SHU	201	001	LEC	Honors Calculus	8:15AM	9:30AM	MW	PDNG	209

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Instructor Schedule

ID 10879463 Charles Newman

Term 1184 Spring 2018

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
17409	MATH-SHU	328	001	LEC	Honors Analysis I	11:15AM	12:30PM	MW	PDNG	304

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林芳华教授授课记录一栏

Fang-Hua Lin	Fall 2013	MATH-SHU	Analysis I
Fang-Hua Lin	Spring 2014	MATH-SHU	Analysis II
Fang-Hua Lin	Spring 2014	MATH-SHU	Analysis II
Fang-Hua Lin	Fall 2014	MATH-SHU	Functions of a Complex Variable
Fang-Hua Lin	Spring 2015	MATH-SHU	Partial Differential Equations
Fang-Hua Lin	Spring 2017	MATH-SHU	Honors Analysis I
Fang-Hua Lin	Spring 2017	MATH-SHU	Independent Study: Mathematics

Instructor Schedule

ID 10625876 Fang-Hua Lin

Term 1138 Fall 2013

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
20497	MATH-SHU	201	001	LEC	Analysis I	11:30AM	12:45PM	TuTh	GEOG	G153

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Instructor Schedule

ID 10625876 Fang-Hua Lin

Term 1144 Spring 2014

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
14678	MATH-GA	2620	002	LEC	Adv Tpcs in Pde:	9:00AM	10:50AM	TuTh	Cour Inst/	517
20295	MATH-SHU	202	001	LEC	Analysis II	1:00PM	2:15PM	MW	GEOG	G149
20296	MATH-SHU	202	002	RCT	Analysis II	1:00PM	2:15PM	F	GEOG	G149

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Instructor Schedule

ID 10625876 Fang-Hua Lin

Term 1148 Fall 2014

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
19199	MATH-SHU	282	001	LEC	Functions of a Complex Variabl	9:45AM	11:00AM	MW	PDNG	304

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Instructor Schedule

ID 10625876 Fang-Hua Lin

Term 1154 Spring 2015

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
5524	MATH-GA	2660	003	LEC	Adv Tpcs in Analysis:	9:00AM	10:50AM	MW	Cour Inst/	517
23391	MATH-SHU	263	001	LEC	Partial Differential Equations	9:45AM	11:00AM	MW	PDNG	201

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Instructor Schedule

ID 10625876 Fang-Hua Lin

Term 1174 Spring 2017

Instructor Schedule | Instructor Schedule 2

Class Number	Subject	Catalog	Section	Component	Class Title	Start Time	End Time	Meeting Days	Building	Room
2889	MATH-GA	2660	002	LEC	Adv Tpcs in Analysis:	1:25PM	3:15PM	TuTh	Cour Inst/	517
13909	MATH-SHU	328	001	LEC	Honors Analysis I	11:15AM	12:30PM	MW	PDNG	212
22212	MATH-SHU	997	001	IND	Independent Study: Mathematics					

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附件 7

上海纽约大学数学与应用数学(荣誉数学方向)
专业本科培养方案

一、 专业简介

数学是科学的基石，它既是一门纯科学，又是解决其他学科问题以及基于现象建模的工具，具有强大的双重作用。因为数学，我们得以在计算中建立有效算法，研究金融市场的罕见事件，模拟物理世界，发展气候科学预测，绘制和研究人类基因组，以及分析人类大脑结构。数学从自然界存在与产生的各种问题，以及工业科技领域的应用中汲取生命力，以严谨和抽象为基石。

二、 培养目标

上海纽约大学的本科教育突破传统意义的专业边界，充分借鉴世界一流高校的人才培养模式，将通识教育培养与专业训练的优势相结合，培养具有创新能力和全球胜任力的未来全球领军人才。

学校于 2012 年开设数学与应用数学专业，分为数学与荣誉数学两个专业培养方向。荣誉数学充分借鉴纽约大学久负盛名的同名专业培养经验，对标基础拔尖人才，瞄准学科理论的前沿和科技社会发展的重大需求，重视培育学生的创造性、创新能力和学术志趣，为数学学科的发展培养高级研究人才预备队的同时，也培养学生的开阔学科视野，塑造全球胜任力和解决复杂问题的应用能力。

毕业生具有扎实的数学和统计基础、熟练的数据分析能力和极高的数学素养，文理并茂，全面发展。绝大多数进入世界著名大学和研究机构攻读数学、应用数学、计算机、信息科学、金融与管理科学等领域的硕士及博士学位。就业方面，毕业生计算机、金融以及大数据等领域具备强大的职业核心竞争力。

三、 授予学位及毕业学分要求

1. 修业年限： 四年
2. 授予学位： 上海纽约大学理学学士学位，纽约大学理学学士学位
3. 最低学分要求： 需修满 128 个学分，学分绩点（GPA）累计 3.65 或以上，且专业课成绩达到 3.65 或以上[1]

[1] 上海纽约大学学分绩点满分为 4.0。

修读荣誉数学方向的学生，需具备扎实的数理基础，并对数学及相关领域拥有浓厚的学术兴趣。在选拔机制上，专业培养不设名额限制。学生在满足前序课程的修课要求后，即可选择荣誉数学方向。在毕业要求上，学生学分绩点（GPA）累计 3.65 或以上，且专业课绩点达到 3.65 或以上，才能以荣誉数学专业毕业。

四、 主要课程设置

上海纽约大学秉承博雅教育的理念，顺应经济全球化及科技高速发展迭代的时代特征，贯彻宽口径、厚基础的原则，将通识培养与专业训练的优势相融合，打造独具特色的本科教育模式。课程设置以一套创新型的“21 世纪通识教育”核心课程为基石，构建以通识教育核心课程、专业课程和通识教育选修课程三大板块为基础的课程体系。

本科阶段前两年实行覆盖所有学生的通识教育，以培养学生全方位的知识、技能和品德。通识教育核心课程体系是一套精心设计，融入相关专业核心课程设置要求中最精华部分，且兼顾多种学科门类的课程体系。具体包括社会和文化基础、写作、数学、自然科学、算法思维、语言六大课程模块。核心课程旨在着重培养五大基础能力，包括社会感知力、思辨分析能力、算法思维、跨文化沟通能力及创造力，为学生面向全球化竞争和科技高速发展带来的挑战赋能。

专业课程的设置经过反复论证，在课程质量标准、课程结构、学业支持体系建设、教学资源配置等多方面充分借鉴纽约大学的机制与做法，严格把握标准。专业培养上，充分利用纽约大学全球体系内丰富优质的教育资源，在培养学生专业素养的同时，也注重拓展学生的国际视野和学术品味。学校所有本科生均需在就读期间，前往纽约大学其他 13 个具有学位授予资格的校园或海外学习中心学习一至二学期。荣誉数学专业的学生通常在大三时期，前往纽约大学库朗数学研究所学习一年，与数学及相关应用领域最顶尖的学者深度交流与学习。库朗所作为全球顶尖的应用数学中心，不仅培养了众多国际数学一流学者，更是培养华人数学家的摇篮。丘成桐、林芳华、田刚、陈贵强、张恭庆、鄂维南等顶尖华人数学家都曾在库朗研究所长期学习和工作过。

数学学科的师资建设成果显著，已初步建成一只国际化、高水平、梯队合理的教师队伍。2021 年，数学常任教师共有 30 位，其中包括 2 位美国院士、2 位高层次人才计划入选者等顶尖科学家，及多位曾在库朗数学研究所有任教及科研经历的青年优秀学者。来自数学学科发展成熟的法国和美国一流数学科研院所的教师占多数。他们深厚的学术功底，前沿的学科视角和丰富的教学经验，有助于培育学生的数学思维和学术视野，培养学生扎实、高水平的学术能力。

荣誉数学专业培养的具体课程设置，包含以下三大课程板块：

（一）通识教育核心课程

通识教育核心课程体系是上纽大博雅教育的基石，方案的制定吸取了纽约大学成熟的本科博雅教育培养理念，融入了各个相关专业不同核心课程设置要求中的精华部分最，并在充分考虑学校的办学使命和定位后，经历了不断的调整和改进。在目前的课程体系设置中，每个学生都会在确定专业选择前，接受社会科学、人文学科、自然科学和数学在内的六大模块核心课程板块的训练。六大核心模块包括：**社会和文化基础、写作、数学、科学、算法思维、及语言**。

有意选择荣誉数学方向的学生只有在完成数学模块**荣誉微积分**相关课程及自然科学模块中**物理、化学及生物课**等相关课程的基础上，并在大一取得优异课业成绩，才能最终选定荣誉数学专业。

（二）专业课程

1. **专业主修课程（32 学分）**：包含一系列课程内容具有挑战度的荣誉课程，涵盖荣誉线性代数、概率论、复变函数，分析学、抽象代数、及荣誉常微分方程等 8 门课程；
2. **专业选修课程（20 学分）**：涵盖功能分析，数理统计，金融数学，数值分析，偏微方程式，实际变量，随机过程导论，拓扑学等高阶课程；
3. **毕业论文（4 学分）**

（三）通识教育选修课程

除通识教育核心课程及专业课程外，通识教育选修课程板块提供大量选修课程供学生进一步选择，以期最大可能拓宽学生的整体知识面，提高其科学、人文、社科、艺术等方面的综合素质。

五、 教学计划及课程选择（毕业要求 128 个学分）

1. 通识教育核心课程列表（部分）：

学生需完成最多 52 学分，占比总学分 40.6%

序号	核心模块（学分）	子模块/课程系列	课程名称	学分	编号
1	社会与文化基础 Social and Cultural Foundation（12-16 学分）	全球视野中的社会（4 学分）Global Perspective on Society	全球视野中的社会 Global Perspective on Society	4	CSCI-SHU 101
		人文视角系列（选一门，4 学分）[2] Perspective on Humanities	人文视角 Perspective on Humanities	4	CCCF-SHU 101
			人文视角专题:超越自然 Perspectives on the Humanities: Beyond Nature	4	CCCF-SHU 101W1
			人文视角专题：语言，身份与世界中的英语 Perspectives on the Humanities: Language, Identity, and World Englishes	4	CCCF-SHU 101W7
			人文视角专题：文学及文学批评 Perspectives on the Humanities: Literature and Its Critics	4	CCCF-SHU 101W17
			人文视角专题：中西文学交流 Perspectives on the Humanities: Sino-Western Literary Exchanges	4	CCCF-SHU 101W21
			人文视角专题：人文医学 Perspectives on the Humanities: Medicine and Disease in the Humanities	4	CCCF-SHU 101W24

[2] “人文视角”系列同时满足《社会与文化基础》及《写作》两大课程模块的修课需求。完成该系列课程中的一门课程即为完成这两大模块的修课需求。系列课程共设 12 个专题课程，由于篇幅有限，本处仅列举其中 6 个专题课程信息。

			人文视角专题:艺术史 Perspectives on the Humanities: Art Histories	4	CCCF-SHU 101W32
		跨学科视角下的中国系列[3]（选两门，8 学分）[4] Interdisciplinary Perspectives on China	现代经济发展史 History of Modern Economic Growth: Exploring China From a Comparative Perspective	4	ECON-SHU 238
			中国环境研究 Chinese Environmental Studies	4	GCHN-SHU 243
			20 世纪东亚美国关系 20th-Century East Asia-U.S. Relations	4	GCHN-SHU 252
			1840 年后的现代中国史 History of Modern China Since 1840	4	HIST-SHU 153
			五千年中国史 5000 Years of Chinese History: Fact or Fiction?	4	HIST-SHU 226
			全球视野下的中国现代史 History of Modern China in a Global Context	4	HIST-SHU 179
			虚构文学中的上海历史与文化 Shanghai Stories	4	HUMN-SHU 366
			认识中国 Concept of China	4	GCHN-SHU 110
			中美关系 US-China Relations	4	SOCS-SHU 275
			在发展中国家投资 International Investment Transactions in Developing Countries: China, Africa, Latin America		LWSO-SHU 491

[3] “跨学科视角下的中国课程”系列涵盖多个学科领域，包括历史、哲学、文化、艺术及文学等。课程教学以多学科比较性的视角出发，旨在培养学生批判性和创新性分析的技能。
[4] “跨学科视角下的中国课程”系列共设 40 门课，由于篇幅有限，本处仅列举其中 10 门课程信息。

			视觉艺术导论：当代艺术中的中国传统方法 Introduction to Studio Art - Chinese Traditional Methods in Contemporary Art	4	ART-SHU 210
			摄影导论 Introduction to Photography I	4	ART-SHU9301
			中国现代作家 Chinese Modern Writers	4	GCHN-SHU 263
			新闻学 Methods and Practice: Journalism	4	JOUR-SHU9202
			当代艺术&新媒体 Contemporary Art & New Media	4	CCCF-SHU 128
			中国新闻与社会 Journalism and Society in China	4	CCCF-SHU 133
			全球中国研究专题课 Topics in Global China Studies	4	GCHN-SHU 200
			全球媒体研讨课 Global Media Seminar: China	4	MCC-SHU9451
			中国的宗教和社会 Religion and Society in China: Ghosts, Gods, Buddhas and Ancestors.	4	RELS-SHU9270
			全球化：上海 Global Connections: Shanghai	4	SCA-SHU9634
2	**写作 （4-8 学分）**	探究性写作 （选一门，4 学分） Writing as Inquiry	探究性写作 I Writing as Inquiry I	4	WRIT-SHU 101
			探究性写作 II Writing as Inquiry II	4	WRIT-SHU 102
			人文视角专题:超越自然 Perspectives on the Humanities: Beyond Nature	4	CCCF-SHU 101W1

		人文视角系列（选一门）[5] Perspectives on Humanities	人文视角专题：语言，身份与世界中的英语 Perspectives on the Humanities: Language, Identity, and World Englishes	4	CCCF-SHU 101W7
			人文视角专题：文学及文学批评 Perspectives on the Humanities: Literature and Its Critics	4	CCCF-SHU 101W17
			人文视角专题：中西文学交流 Perspectives on the Humanities: Sino-Western Literary Exchanges	4	CCCF-SHU 101W21
			人文视角专题：人文医学 Perspectives on the Humanities: Medicine and Disease in the Humanities	4	CCCF-SHU 101W24
			人文视角专题:艺术史 Perspectives on the Humanities: Art Histories	4	CCCF-SHU 101W32
3	**数学**[6]（**4 学分**）	数学（选一门）Mathematics	微积分预修 Precalculus	4	MATH-SHU 9
			定量推理 Quantitative Reasoning: Great Ideas in Mathematics	4	MATH-SHU 10
			微积分 Calculus	4	MATH-SHU 121
			线性代数 Linear Algebra	4	MATH-SHU 140
			网络与动力学 Networks and Dynamics	4	MATH-SHU 160
			荣誉微积分 Honors Calculus	4	MATH-SHU 201

[5]“人文视角”系列同时满足社会与文化基础及写作两大课程模块的修课需求。

[6] 数学课程的要求或因专业或者学生具体情况而不同。荣誉数学方向的学生必须通过“数学”核心模块的考核要求。

			概率概论 Honors Theory of Probability	4	MATH-SHU 233
4	**自然科学**[7] **（8 学分，在任意两个子模块选）**	通过实验发现自然世界系列（选一门）[8] Experimental Discovery in the Natural World	科学基础之生物实验 Foundation of Science: Biology Laboratory	2	BIOL-SHU 123
			化学基础 II Foundations of Chemistry II	3	CHEM-SHU 126
			科学基础之化学实验 Foundation of Science: Chemistry Laboratory	2	CHEM-SHU 127
			通用物理 I General Physics I	3	PHYS-SHU 11
			物理实验 Foundation of Science: Physics Laboratory	2	PHYS-SHU 71
			荣誉基础物理 I Foundations of Physics I	3	PHYS-SHU 91
			光学成像 Optical Imaging: Applications in Biology and Engineering	2	PHYS-SHU 200
			量子理论学 Topics: Introduction to Quantum Theory and Technology	2	PHYS-SHU 201
		科学，技术与社会系列（选一门）[9] Science, Technology, and Society	创新导论 Topics: Creativity Considered	4	CCST-SHU 132
			水的历史 History of Water	4	HIST-SHU 302
			研讨会主题：社会媒体的政治性使用 Seminar Topics: Political Uses of Social Media	4	INTM-SHU 295

[7] 选择荣誉数学专业的学生必须修读“自然科学”模块中荣誉基础物理，基础生物及基础化学其中的两门课，并完成对应其中一门课的试验环节部分。

[8]“通过实验发现自然世界”系列课程共开 15 门课，由于篇幅有限，本处仅列举其中 10 门课程信息。

[9]“科学，技术与社会”系列课程共开 31 门课，由于篇幅有限，本处仅列举其中 10 门课程信息。

			速度与语言的神经基础 Neural Bases of Speech and Language	4	NEUR-SHU 265
			自由意志与大脑 What Can Neuroscience Tell us about Free Will	4	NEUR-SHU 10
			技术哲学：思维机器 Philosophy of Technology: Thinking Machines	4	PHIL-SHU 130
			环境与社会 Environment and Society	4	SOCS-SHU 135
			瘟疫学 Pestilence: Critical Perspectives in Global Health	4	SOCS-SHU 306
			全球环境政治学 Global Environmental Politics	4	SOCS-SHU 333
		算法思维课程系列（选一门） Algorithmic Thinking Courses	数位逻辑 Digital Logic	4	CENG-SHU 201
			计算机编程概论 Introduction to Computer Programming	4	CSCI-SHU 11
			计算机科学概论 Introduction to Computer Science	4	CSCI-SHU 101
			数据结构 Data Structures	4	CSCI-SHU 210
			通信实验室 Communications Lab	4	INTM-SHU 120
5	**算法思维（4 学分）**	算法思维课程系列（选一门） Algorithmic Thinking Courses	数位逻辑 Digital Logic	4	CENG-SHU 201
			计算机编程概论 Introduction to Computer Programming	4	CSCI-SHU 11
			计算机科学概论 Introduction to Computer Science	4	CSCI-SHU 101
			数据结构 Data Structures	4	CSCI-SHU 210

			通信实验室 Communications Lab	4	INTM-SHU 120
			数值分析 Numerical Analysis	4	MATH-SHU 252
6	**语言**[10] **（8-16 学分）**	汉语课程 （16 学分，选四门） Chinese Language Courses	初级汉语 I Elementary Chinese I	4	CHIN-SHU 101
			初级汉语 I（仅限物理、化学、生物、神经科学专业的学生） Elementary Chinese I - Foundation of Science 1	2	CHIN-SHU 101S
			初级汉语 II（仅限物理、化学、生物、神经科学专业的学生） Elementary Chinese I - Foundation of Science 2	2	CHIN-SHU 101S2
			初级汉语高阶课程 Elementary Chinese I for Advanced Beginners	4	CHIN-SHU 111
			中级汉语 I Intermediate Chinese I	4	CHIN-SHU 201
			中级汉语 I-强化 Intermediate Chinese I - Accelerated	4	CHIN-SHU 201A
			中级汉语 II Intermediate Chinese II	4	CHIN-SHU 202
			中级汉语 II - 强化 Intermediate Chinese II - Accelerated	4	CHIN-SHU 202A
			中级汉语高阶课程 Intermediate Chinese I for Advanced Beginners	4	CHIN-SHU 211
			高级汉语 I Advanced Chinese I	4	CHIN-SHU 301
			高级汉语 II Advanced Chinese II	4	CHIN-SHU 302

[10] 语言核心模块课程中的汉语及英语课程要求因学生语言水平而异。一般而言，非英语母语的学生须完成 8 学分的《学术英语》课程学习；非汉语母语的学生须完成最少 16 学分的汉语课程学习。

			汉语文言文 I Classical Chinese I	4	CHIN-SHU 401
			汉语文言文 II Classical Chinese II	4	CHIN-SHU 402
			中国文化读本 I Readings in Chinese Culture I	4	CHIN-SHU 403
			中国文化读本 II Readings in Chinese Culture II	4	CHIN-SHU 404
			当代中国概况 I Introduction to Contemporary China I	4	CHIN-SHU 415
			当代中国概况 II Introduction to Contemporary China II	4	CHIN-SHU 416
			中国商务与财经 Chinese Business and Finance -- A Bilingual Introduction	4	CHIN-SHU 429
		英语课程 （8 学分，选两门） English Language Courses	学术英语 I English for Academic Purposes I	4	ENGL-SHU 100
			学术英语 II English for Academic Purposes II	4	ENGL-SHU 102

2. **专业必修课程列表（部分）：**

学生需完成 56 学分，占比总学分 43.8%

序号	专业模块	课程名称（中英文）	学分	编号
1	专业主修 （32 学分，选八门）Required Mathematics Courses	荣誉线性代数 I Honors Linear Algebra I	4	MATH-SHU 141
		荣誉线性代数 II Honors Linear Algebra II	4	MATH-SHU 142
		荣誉概率论 Honors Theory of Probability	4	MATH-SHU 233

		复变函数 Functions of a Complex Variable	4	MATH-SHU 282
		荣誉分析学 I Honors Analysis I	4	MATH-SHU 328
		荣誉分析学 II Honors Analysis II	4	MATH-SHU 329
		荣誉代数 I 或 Honors Algebra I or	4	MATH-SHU 348
		微分几何 Differential Geometry	4	MATH-SHU 377
		荣誉常微分方程 Honors Ordinary Differential Equations	4	MATH-SHU 362
2	专业选修[11] (20 学分，选五门) Math Electives	网络和动力学 Networks and Dynamics	4	MATH-SHU 160
		功能分析 Functional Analysis	4	MATH-SHU 226
		数理统计 Mathematical Statistics	4	MATH-SHU 234
		金融数学 Mathematics of Finance	4	MATH-SHU 250
		数学建模导论 Introduction to Math Modeling	4	MATH-SHU 250
		数值分析 Numerical Analysis	4	MATH-SHU 252
		偏微分方程式 Partial Differential Equation	4	MATH-SHU 263
		实际变量 Real Variables	4	MATH-SHU 339
		随机过程学导论 Introduction to Stochastic Processes	4	MATH-SHU 345
		荣誉代数 II Honors Algebra II	4	MATH-SHU 349
		拓扑学 Topology	4	MATH-SHU 375
3	毕业论文 （4 学分）	毕业论文 Senior Thesis	4	n/a

[11] 数学“专业选修”课程共开 17 门课，由于篇幅有限，本处仅列举其中 10 门课程信息。

3. 通识教育选修课（部分）：

学生需完成至少 20 学分，占比总学分 15.6%

在满足通识教育核心课程和专业必修课程学分要求之外，学生跨专业、跨学科领域修读的其他课程均可计为通识选修课程。以 2019-2020 学年为例，依托丰富的课程资源，全校本科课程开设数量为 460 门，平均每门课注册学生人数在 12 人左右。充分保障了互动式、小班化教学的教学方式。

附录：数学及应用数学（荣誉数学方向）教学安排样本

一年级				
	课程 1	**课程 2**	**课程 3**	**课程 4**
秋季学期	通识教育核心课[12]（全球视野中的社会）	通识教育核心课（荣誉微积分）	专业必修课（荣誉线性代数 I）	英语或中文/通识教育核心课/通识教育选修课
春季学期	通识教育核心课（写作）	专业必修课（荣誉数学分析 I ）	专业必修课（荣誉线性代数 II）	英语或中文/通识教育核心课/通识教育选修课
二年级				
	课程 1	**课程 2**	**课程 3**	**课程 4**
秋季学期	通识教育核心课人文视角	专业必修课（荣誉数学分析 II）	专业必修课（荣誉常微分方程）	通识教育核心课/通识教育选修课/中文
春季学期	通识教育核心课	专业必修课（复变函数）	专业必修课（荣誉概率论）	通识教育核心课/通识教育选修课/中文
三年级				
	课程 1	**课程 2**	**课程 3**	**课程 4**
秋季学期	通识教育核心课/通识教育选修课	专业选修课	专业选修课/或通识教育选修课	通识教育必修课/通识教育选修课
春季学期	通识核心课/通识选修课	专业选修课	专业选修课/或通识教育选修课	通识教育选修课
四年级				
	课程 1	**课程 2**	**课程 3**	**课程 4**
秋季学期	通识教育选修课	专业必修课（荣誉代数）	专业选修课/或通识教育选修课	通识教育选修课
春季学期	通识教育选修课	专业选修课	专业选修课/或通识教育选修课	通识教育选修课

[12] 荣誉数学基础课、主修课及选修课标记为黄色。