Impact of Automation on Labor Market and Income Inequality

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Abstract

There are rising concerns over the displacement of human labor by machines. This paper will examine automation in a framework where capital plays an increasingly important role in production in terms of productivity and intensity, respectively. The model features dynamics between heterogeneous workers and two sectors with different elasticity of substitution. We show that with proper conditions, under proposed automation, the employment share between two sectors and income inequality stabilizes in the long run, enabling the coexistence of labor and capital in existing tasks without the need of creating new tasks to replace those already automated ones.

1 Introduction

Automation has been an increasingly hot topic with the recent resurgence of artificial intelligence. Workers are most concerned that they will be displaced by the more and more competent equipment. And this concern is backed up by, for example, Autor (2015), where he argues that computers are capable of substituting for workers in performing routine tasks. Moreover, suggested by Karabarbounis and Neiman (2013), recent statistics on employment to population ratio in the US that show a decline in global labor share can be attributed to firms shifting away from labor and towards capital, which indicates that the displacement effect may have already begun. Unfortunately, the effect of displacement of labor may not be limited to routine jobs. As shown by Beaudry et al. (2016) and Valletta (2018), with lower skill jobs being displaced, college wage premium starts to flatten in recent years and higher skill workers are forced to move down the occupational ladder to work in less skill-intensive jobs.

Despite these possible consequences of automation in mind, we are still unclear of and there is no consensus of which form automation takes mathematically to enable theoretical analysis. Thus, we take a step forward in suggesting two types of automation and analyze its long-term implications on labor market and income distribution. As pointed out by Autor and Dorn (2013) and Autor (2015), automation has a strongly biased impact on labor market in that it mainly displaces jobs with routine tasks. Thus, our model consists of two sectors, service and manufacturing, and heterogeneous workers with continuous skill distribution to take into consideration of the biased impact of automation. Furthermore, over the last few decades, stock of information processing equipment has dramatically increased, as shown in figure 1. This increased stock suggests a higher productivity of capital as a whole in production, meaning that capital might have been used more intensively relative to labor. This feature motivates us to include intensity of capital and productivity of capital (i.e. the share parameter in CES production function) as two factors of automation.

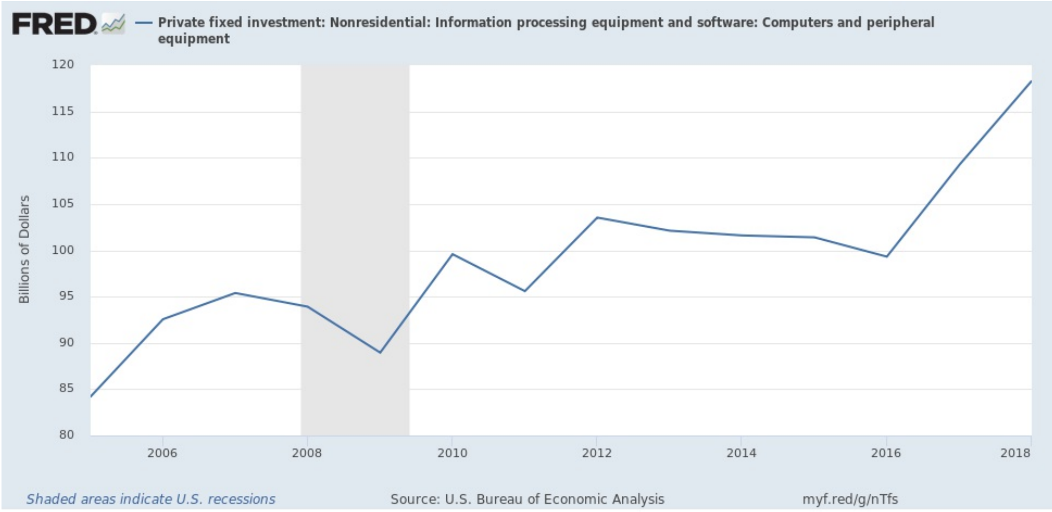

Figure 1. Investment in Information Processing Equipment

While Acemoglu and Restrepo (2017) proposes an theoretical framework that is aimed to find a stable growth path between capital and labor, where capital does not dominate the production and drive out labor. They did so by enabling new tasks to be created, over which human labor has comparative advantage. Creation of new tasks is certainly a way to offset the automation of jobs that human labor no longer has comparative advantage over. However, this also indicates that the entire pool of jobs will be replaced by new tasks over time, which leaves no room for labor to coexist with capital in certain tasks that may not be able to be replaced by new ones. Thus, to complement the existing literature on automation, the automation our model proposes will enable labor to coexist with capital in the long run, and it also features a stationary income inequality in the long term.

We start with a baseline model in section 2 that makes clear the setup of model. There are two sectors, service sector and manufacturing sector, in which labor and capital are gross complements and gross substitutes respectively. Heterogeneous workers with continuous skill distribution are matched with firms in a one-to-one positive assignment way. Following the model setup, we propose the conditions for equilibrium and then derive it analytically. In section 3, two types of automation are proposed and defined mathematically, of which the new equilibria are then derived. Profit, wage and employment share of these two types of automation are at the core of the rest of section 3. In section 4, we focus on income inequality, Gini coefficient, top income inequality and Lorenz curve to be specific. We then discover the long-run non-trivial asymptotic behavior of one type of the automation. Section 5 concludes the paper.

2 The Model

2.1 Household

There is a unit measure of continuum of agents with skill level $L \in [\underline{L}, \overline{L}]$, associated with distribution function $F(L)$.

Consumers purchase a composite good Y, which is defined as a combination of the aggregation of two differentiated goods, denoted as Y_s and Y_g, produced by service sector and manufacturing sector respectively. The composite good Y is composed in a Cobb-Douglas fashion[1]:

$$Y = Y_s Y_g \tag{1}$$

A representative household maximize their utility $u(\cdot)$ with $u'(\cdot) > 0$ and $u''(\cdot) < 0$:

$$\max_{Y} \quad u(Y) \tag{2}$$

which is equivalent to

$$\max_{Y_s, Y_g} \qquad Y = Y_s Y_g \tag{3}$$

$$\text{subject to}$$

$$\begin{cases} \sum_{i \in \{s,g\}} P_i Y_i = P_Y Y \\ P_Y = 1 \end{cases} \tag{4}$$

We treat the price of the composite good as the numeraire. Note that the budget constraint (income $= P_Y Y$) for household suggests all wage as well as profit of firms go to households in the end, through dividends for example. With heterogeneous workers, we shall make a special assumption on household to justify this: similar to Lucas (1990), we view consumers and firm owners as members of a single family that pool their resources and make collective decisions to purchase the composite goods in the end of the day.

Solving the maximization problem above yields the price and quantity relationships:

$$P_s = \frac{1}{2} Y_g \quad \text{and} \quad P_s = \frac{1}{2} Y_s \tag{5}$$

2.2 Firm

There are a unit measure of continuum of firms with proprietary endowment capital K, $K \in [\underline{K}, \overline{K}]$, which can also be viewed as an indicator of quality, the higher the better. The distribution function of K is denoted as $G(K)$, $G(K) \in C^2([\underline{K}, \overline{K}])$, with density function $g(K)$.

Firms produce using the following CES technology:[2]

$$y_i = \left(K^{\frac{\rho_i - 1}{\rho_i}} + L^{\frac{\rho_i - 1}{\rho_i}} \right)^{\frac{\rho_i}{\rho_i - 1}}, i \in \{s, g\} \tag{6}$$

1. For simplicity and without loss of generality, we set both share parameters to be 1. Share parameters are not in the interest of this paper. But one can easily relax this assumption and make them variables despite complications.

Given the fact that manufacturing sector requires relatively more capital stock to start with, which acts as an increased barrier and burden for firms to enter the market, we impose an exogenous *entry cost* c for firms in manufacturing sector. Furthermore, as categorized by Autor and Dorn (2013), we assume manufacturing sector uses labor and capital more as substitutes, given the more routine nature of the tasks, compared with the service sector. Thus, we impose that labor and capital are gross complements in service sector and are gross substitutes in goods sector, which requires that $\rho_s \in (0,1)$ and $\rho_g \in (1,+\infty)$.

2.3 One-to-one Matching

For simplicity and in order to keep model tractable, we impose an one-to-one matching between workers and firms.

Proposition 1. *Positive assignment in one-to-one matching maximizes the aggregate output.*

Positive assignment simply means best workers are matched with firms that have highest capital level. The complete proof is given by Becker (1932) and a simplified two-firm two-agent case is shown below, which can be extended to the continuous case.

Proof.

Total output with positive assignment - Total output with negative sorting

$=Q(x_1, y_1) - Q(x_1, y_2) + Q(x_2, y_2) - Q(x_2, y_1)$

$=-\int_{y_1}^{y_2} \frac{\partial Q}{\partial y}(x_1, y)\,\mathrm{dy} + \int_{y_1}^{y_2} \frac{\partial Q}{\partial y}(x_2, y)\,\mathrm{dy}$

$=\int_{y_1}^{y_2} \left[\frac{\partial Q}{\partial y}(x_2, y) - \frac{\partial Q}{\partial y}(x_1, y)\right] \mathrm{dy}$

$=\int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial^2 Q}{\partial x \partial y}(x, y)\,\mathrm{dxdy}$

which is positive when $\frac{\partial^2 Q}{\partial x \partial y} > 0$ and negative when $\frac{\partial^2 Q}{\partial x \partial y} < 0$. □

In our case,

$$\frac{\partial^2 Q}{\partial x \partial y} \equiv \frac{\partial^2 y_i(K, L)}{\partial K \partial L} = 1 > 0$$

which means in our model, we should follow positive assignment to maximize the aggregate output.

2.4 Equilibrium

The Threshold Capital Level:

2. For simplicity, we set both share parameters to be 1 in the production function in the baseline model to present a cleaner version of equilibrium. This assumption will be relaxed in section 3.

With the observation that service sector generally uses less capital relative to the manufacturing sector, for example, a restaurant versus a shoe factory, we introduce a threshold capital level $K_* \in (\underline{K}, \overline{K})$ which separates the two sectors. Thus, firms with $K \in [\underline{K}, K_*]$ belong to service sector and firms with $K \in [K_*, \overline{K}]$ belong to manufacturing sector.

2.4.1 Special Case - Uniform Distribution of L and K

From now on, for simplicity and tractability of the model, we assume $F \sim U[a, b]$ and $G \sim U[a, b]$.

Equilibrium conditions:

***1. Labor Market Clearing Conditions*:**

We impose that workers' outside options are zero so that workers strictly prefer to work as long as they receive a positive wage. Since there is a unit measure of firms and workers, one-to-one matching ensures full employment. Positive sorting requires equating left/right tales of the distribution. Let $\alpha(K)$ denote the assignment function such that $\alpha(K)$ gives the value of skill of the matched worker to firm with capital K.

$$1 - F(\alpha(K)) = 1 - G(K) \tag{7}$$

Since $F \sim U[a, b]$ and $G \sim U[a, b]$, equation (7) gives:

$$L = \alpha(K) = K \tag{8}$$

Therefore, $K_* = L_*$, i.e. the marginal worker L_* between two sectors has skill level K_* and, in addition, workers with $L \in [a, L_*]$ are employed by the service sector while those with $L \in (L_*, b]$ are employed by the manufacturing sector.

***2. Profit Maximization*:**

Firms maximize profit and we denote wage for service sector as w_s, and w_g for manufacturing sector:

$$\underset{L}{\text{Max}} \quad P_i \left(K^{\frac{\rho_i - 1}{\rho_i}} + L^{\frac{\rho_i - 1}{\rho_i}} \right)^{\frac{\rho_i}{\rho_i - 1}} - w_i(L) - c \cdot \mathbb{1}_{i=g} \tag{9}$$

where $i \in \{s, g\}$ and c is the exogenous constant entry cost for firms in the manufacturing sector.

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$$P_i \left(K^{\frac{\rho_i - 1}{\rho_i}} + L^{\frac{\rho_i - 1}{\rho_i}} \right)^{\frac{1}{\rho_i - 1}} L^{-\frac{1}{\rho_i}} = \frac{d}{dL} w_i(L) \tag{10}$$

$$\Rightarrow \qquad 2^{\frac{1}{\rho_i - 1}} P_i = \frac{d}{dL} w_i(L) \tag{11}$$

Solve the ODE in equation (11) with initial condition $w_s(a) = 0$[3], while we do not restrict the initial condition for w_g:[4]

$$w(L) = \begin{cases} 2^{\frac{1}{\rho_s - 1}} P_s\,(L - a) & \text{for } i = s \text{ and } L \in [a, L_*] \\ 2^{\frac{1}{\rho_g - 1}} P_g\,(L - a) + \mathcal{C} & \text{for } i = g \text{ and } L \in (L_*, b] \end{cases} \tag{12}$$

where $\mathcal{C}$ is an arbitrary constant to be determined endogenously, $i \in \{s, g\}$, $L \in [a, L_*]$ for $i = s$ and $L \in (L_*, b]$ for $i = g$.

Equation (12) can be rewritten as:

$$w_i(L) = 2^{\frac{1}{\rho_i - 1}} P_i\,(L - a) + \mathcal{C} \cdot \mathbb{1}_{i=g} \tag{13}$$

***3. Firms' Optimal Choice of Sector*:**

In equilibrium, firms optimize their choices of sectors so that it is not profitable to switch to the other sector. From equation (8) and with equilibrium wage (13), we can write equation (9), firms' profit, as:

$$\pi_i(K) = 2^{\frac{1}{\rho_i - 1}} P_i\,(K + a) - (\mathcal{C} + c) \cdot \mathbb{1}_{i=g} \tag{14}$$

which is linear in K under equilibrium. Since profit is linear in K and is monotonic, the optimal choice condition is reduced to there being no gap in profit for the marginal firm K_*. Therefore, the intersection of two sectors' profit function will determine the threshold capital level K_*, which satisfies:

$$2^{\frac{1}{\rho_s - 1}} P_s\,(K_* + a) = 2^{\frac{1}{\rho_g - 1}}\,P_g\,(K_* + a) - (\mathcal{C} + c) \tag{15}$$

4. *Workers' Indifference Condition:*

With mobility between sectors, we need to impose an indifference condition such that the marginal agent $L_* = K_*$ is indifferent between working in both sectors:

$$\text{then} \quad \lim_{L \uparrow L_*^-} w_s(L) = \lim_{L \downarrow L_*^+} w_g(L)$$

Thus, this gives us the initial condition for $w_g(L)$ such that $w_g(L_*) = w_s(L_*) = 2^{\frac{1}{\rho_s - 1}} P_s\,L_*$,

$$\Rightarrow \quad 2^{\frac{1}{\rho_s - 1}} P_s\,(L_* - a) = 2^{\frac{1}{\rho_g - 1}} P_g\,(L_* - a) + \mathcal{C} \tag{16}$$

From equation (5), we also know that:

$$P_s = \frac{1}{2} Y_g = \frac{1}{2} \int_{K_*}^{b} \left(K^{\frac{\rho_g - 1}{\rho_g}} + L^{\frac{\rho_g - 1}{\rho_g}} \right)^{\frac{\rho_g}{\rho_g - 1}} \mathrm{dK} = 2^{\frac{2 - \rho_g}{\rho_g - 1}} (b^2 - K_*^2) \tag{17}$$

$$P_g = \frac{1}{2} Y_s = \frac{1}{2} \int_{a}^{K_*} \left(K^{\frac{\rho_s - 1}{\rho_s}} + L^{\frac{\rho_s - 1}{\rho_s}} \right)^{\frac{\rho_s}{\rho_s - 1}} \mathrm{dK} = 2^{\frac{2 - \rho_s}{\rho_s - 1}} (K_*^2 - a^2) \tag{18}$$

3. Here, by $w(a) = 0$, we assume the worker with the lowest skill does not contribute to the production at all. We can think of it as a skill too obsolete to be useful.

4. Note that the manufacturing sector only hires workers with $L = L_* > a$. And given the entry cost c, $w(a) = 0$ is no longer a necessary initial condition for firms in manufacturing sector, since, for example, hiring a worker with $L = a$ may result in negative profit.

Combine equations (15)(16)(17)(18), we can solve for the equilibrium:

$$
\begin{cases}
\text{Wage: } w_i(L) = 2^{\frac{1}{\rho_i - 1}} P_i (L - a) + \mathcal{C} \cdot \mathbb{1}_{i=g} \\
\text{Profit: } \pi_i(K) = 2^{\frac{1}{\rho_i - 1}} P_i (K + a) - (\mathcal{C} + c) \cdot \mathbb{1}_{i=g} \\
\text{Price of service: } P_s = 2^{\frac{2-\rho_g}{\rho_g - 1}} (b^2 - K_*^2) \\
\text{Price of goods: } P_g = 2^{\frac{2-\rho_s}{\rho_s - 1}} (K_*^2 - a^2) \\
\text{ODE Constant: } \mathcal{C} = \left(2^{\frac{1}{\rho_s - 1}} P_s - 2^{\frac{1}{\rho_g - 1}} P_g \right) (K_* - a) \\
\text{Threshold Capital: } K_* = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}
\end{cases}
$$

where $p = -\frac{a^2 + b^2}{2}$, $q = -c \cdot 2^{\frac{1 - \rho_g \rho_s}{(\rho_g - 1)(\rho_s - 1)}}$, $L \in [a, K_*]$ for $i = s$ and $L \in (K_*, b]$ for $i = g$.

Proof. Appendix 1. □

2.4.2 Properties of the Equilibrium

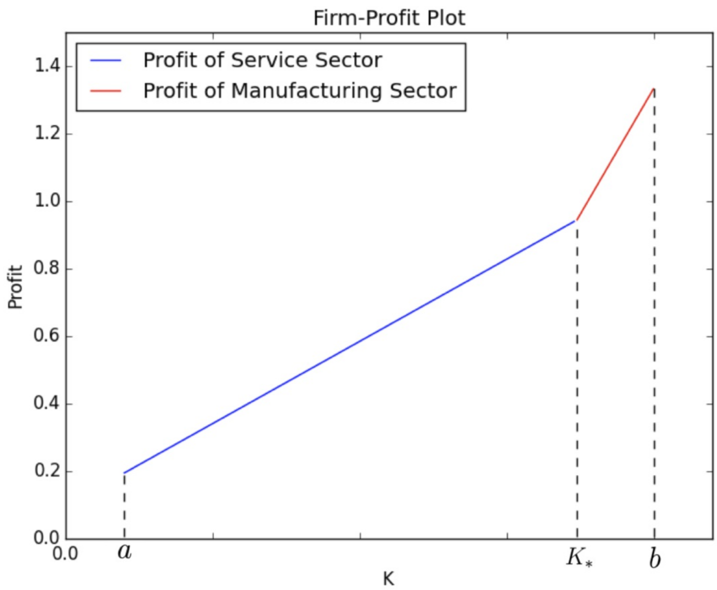

Figure 2. Firm-Profit Plot

Figure 2 shows the profit earned by firms with endowment capital from a to b. It is clear that profit is piecewise with $K \in [a, K_*]$ being the profit function of service sector and $K \in [K_*, b]$ being that of manufacturing sector. The discontunuity at $K = K_*$ indicates the marginal firm that is indifference between two sectors, while firms with higher capital are strictly better off in manufacturing sector.

Property. *(i). K_* is the marginal capital level such that firms with capital level above it earn more in manufacturing sector. And it determines proportion of firms in each sector.*

To make it clearer, we also plot the same profit function with extended lines (Figure 3). It is evident that for firms with capital lower than K_*, they earn lower profit in manufacturing factor than in service sector, and the converse is true for firms with capital higher than K_*. The proportion of firms in service sector is thus $\frac{K_* - a}{b - a}$.

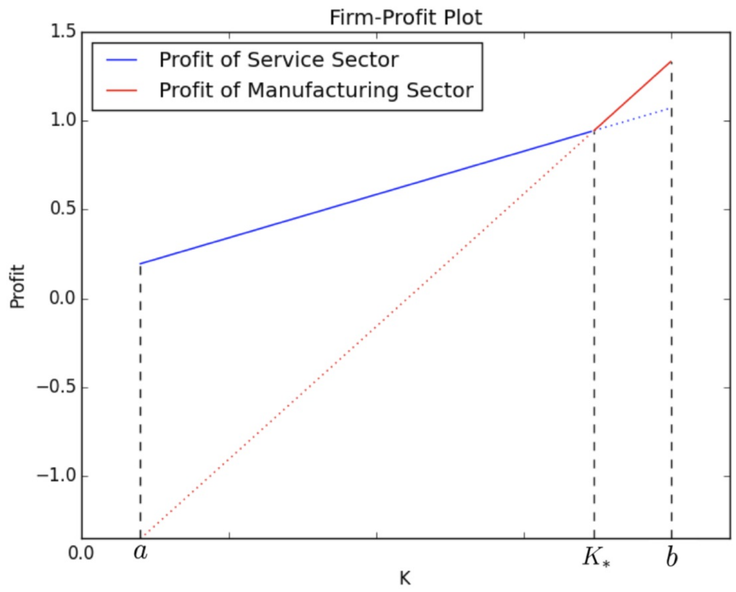

Figure 3. Firm-Profit Plot with Extended Profit Functions

Similar argument can be made regarding wage. Since firms and workers in the model both extract surplus via their respective marginal product of inputs (capital for firms and labor for workers), profit is essentially “wage” for firms. Figure 4 shows the wage function.

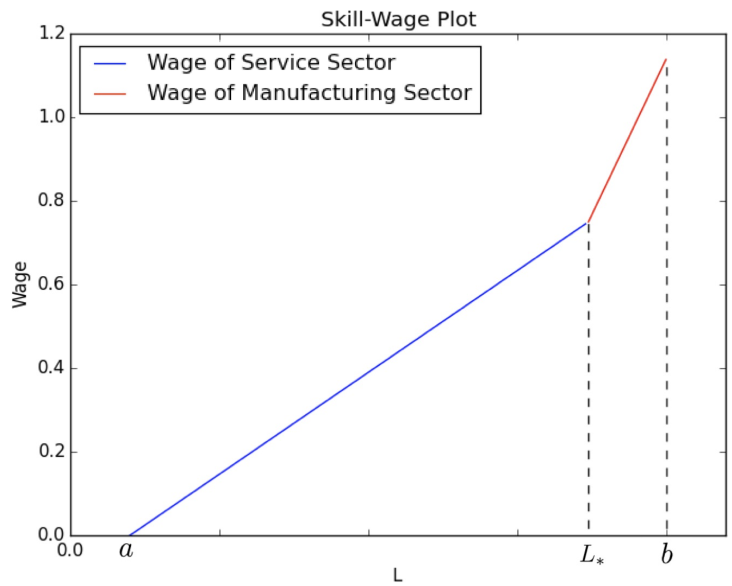

Figure 4. Skill-Wage Plot

Property. *(ii). Profit and wage functions always have same slope due to the symmetry between firms and workers, which is a result from one-to-one matching with positive assignment.*

3 Automation (Comparative Statics)

3.1 Types of Automation

Before jumping into the discussion of two types of automation, we first explore the case which may be of interest to the readers and explain why we did not work out the case. This case can be investigated in future research. We call it type 0. Readers can skip type 0 and start with type 1 directly.

Type 0: Increasing substitutability of labor and capital

Automation under which machine and labor become more of substitutes can be classified mathematically as increases in the elasticity of substitution in the production function. In this case, firms that comprise more routine job, i.e. those in manufacturing sector, suffer from a larger increase in elasticity of substitution than those in service sector, in which the work usually involves abstract task such as human interaction. Krusell et al. (2000) examined the higher substitutability between capital and lower skill labor and Caines et al. (2017) confirmed the higher complementarity between capital and labor in complex tasks. To illustrate why manufacturing sector and service sector consist of different types of tasks, we give the following examples:

An example of work in service sector is servers in restaurants or luxury hotels. This type of business usually possess relatively less capital but more labor compared with manufacturing sector. The tasks in such places could entail providing personalized service to retain customers, sometimes even beyond the scope of their duties. Thus, labor hired by service sector is more resistant to the trend of automation overall.

An example of work in manufacturing sector is shoe manufacturers, who usually work on assembly lines, which could be updated easily such that fewer workers are needed. Even in some more complicated manufacturing tasks that were less prone to automation, such as watch making, with the advancement of computers, these tasks are also being automated. Thus, labor hired by the manufacturing sector is potentially exposed to higher probability of being displaced by machines.

Ideally, the dynamics of this type of automation can be as follows: at initial state, all firms in the same sector are homogeneous in their technology with $\rho_s \in (0, 1)$ and $\rho_g \in (1, +\infty)$. Automation takes form of an increment s_i such that $s_i > 0$. After automation, the production function is as follows:

$$y(i) = \left(\beta K_i^{\frac{\tilde{\rho_i} - 1}{\tilde{\rho_i}}} + (\beta - 1) L_i^{\frac{\tilde{\rho_i} - 1}{\tilde{\rho_i}}}\right)^{\frac{\tilde{\rho_i}}{\tilde{\rho_i} - 1}} \tag{19}$$

where $\tilde{\rho_i} = \rho_i + s_i, i \in \{s, g\}$ and $\tilde{\rho_s} < 1$.

To make it clearer why this case will not work in our model, we solve for the equilibrium explicitly:

$$\begin{cases} \text{Wage: } w_i(L) = (1-\beta)P_i\,(L-a) + \mathcal{C}\cdot \mathbb{1}_{i=g} \\ \text{Profit: } \pi_i(K) = P_i\,[\beta\,K + (1-\beta)a] - (\mathcal{C}+c)\cdot \mathbb{1}_{i=g} \\ \text{Price of service: } P_s = \frac{b^2 - K_*^2}{4} \\ \text{Price of goods: } P_g = \frac{K_*^2 - a^2}{4} \\ \text{ODE Constant: } \mathcal{C} = (1-\beta)(P_s - P_g)(K_* - a) \\ \text{Threshold Capital: } K_* = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \end{cases}$$

where $p = -\frac{a^2+b^2}{2}$, $q = -2c$, $L \in [a, K_*]$ for $i = s$ and $L \in (K_*, b]$ for $i = g$.

Proof. Appendix (4) □

It is evident but also surprising that the equilibrium is **independent of $\boldsymbol{\rho_i}$** in our model. And we explain this result with the following proposition:

Proposition 2. *Output and marginal products of production factors of CES production function are invariant under changes in elasticity of substitution if the matching mechanism is one-to-one positive assignment with same size and distribution of inputs, or more generally, if all inputs are with same amount.*

Proof. We first show that under one-to-one positive assignment and with same size and distribution of labor and capital, the resulting match is $L = K$.

Suppose distributions are $L \sim F(L)$ and $K \sim G(K)$, then positive assignment requires:

$$1 - F(\alpha(K)) = 1 - G(K)$$

Since $F = G$, equation (7) gives:

$$L = \alpha(K) = K$$

1. Invariance of marginal products:

It is easily shown by taking derivative of the production function:

$$\frac{\partial y_i}{\partial L} = (1-\beta)\left(\beta K^{\frac{\rho_i - 1}{\rho_i}} + (1-\beta)L^{\frac{\rho_i-1}{\rho_i}}\right)^{\frac{1}{\rho_i - 1}} L^{-\frac{1}{\rho_i}} = (1-\beta)L$$
$$\frac{\partial y_i}{\partial K} = \beta\left(\beta K^{\frac{\rho_i - 1}{\rho_i}} + (1-\beta)L^{\frac{\rho_i-1}{\rho_i}}\right)^{\frac{1}{\rho_i - 1}} K^{-\frac{1}{\rho_i}} = \beta K$$

2. Invariance of output:

$$y_i = \left(\beta K^{\frac{\rho_i - 1}{\rho_i}} + (1-\beta) L^{\frac{\rho_i - 1}{\rho_i}}\right)^{\frac{\rho_i}{\rho_i - 1}} = K = L$$

□

More generally, as proved by Beckenbach and Bellman (1961)(1961, ?):

The mean of order x of the positive values k_i with weights a_i:

$$\left(\sum_{i=1}^{n} a_i k_i^x\right)^{\frac{1}{x}}$$

is strictly increasing in x, unless all k_i's are equal.

We now explore two types of automation:

Type 1: Increasing productivity of capital

The first type of automation is defined as an increase in the factor productivity of capital. This type of automation is reasonable in that an increase in factor productivity simply means, in a unit of time, more work is done by machines than human. And this certainly matches our expectation of automation. In our model, to mimic the effect of an increase in factor productivity of capital, it will take the form of increasing capital endowment of firms.

Specifically, we will study the production function

$$y_{it} = \left(K_{it}^{\frac{\rho_i - 1}{\rho_i}} + L_{it}^{\frac{\rho_i - 1}{\rho_i}}\right)^{\frac{\rho_i}{\rho_i - 1}} \tag{20}$$

with an increasing K_{it} such that $K_{i(t+1)} = \phi K_{it} \, (\phi > 1) \, \forall i$.

Type 2: Increasing intensity of capital in production

The second type of automation is defined as an increase in the intensity of capital in production, i.e. the share parameter of capital increases. We study this type of automation in order to separate another channel of automation, which is through more reliance on capital in production, but not necessarily with a change in capital itself. This type of automation could take place, for example, when human workers have less incentive to work on certain types of (e.g. boring) tasks and as a result, the intensity of capital involved in production increases.

Specifically, we study the following production function

$$y_i = \left(\beta K^{\frac{\rho_i - 1}{\rho_i}} + (1-\beta) L^{\frac{\rho_i - 1}{\rho_i}}\right)^{\frac{\rho_i}{\rho_i - 1}}, i \in \{s, g\} \tag{21}$$

with an increasing β, where $\beta \in (0, 1)$.

3.2 Type 1: Increasing Productivity of Capital

As a result of automation, it is reasonable to expect an increase in the productivity capital relative to labor in production, since more work is done by machines.

If we assume a constant growth rate of capital for all firms such that $K_{t+1} = \phi K_t \, (\phi > 1)$, then the probability distribution function $G(K)$ and its density function $g(K)$ will have the following property:

$$G_t(K) = G_0\left(\frac{K}{\phi^t}\right) \quad \text{and} \quad g_t(K) = \frac{1}{\phi^t} g_0\left(\frac{K}{\phi^t}\right) \tag{22}$$

Proof. Appendix 2. □

The case of uniform distribution:

In the case where $G_0(K) \sim U[a, b]$, we have $G_t(K) \sim U[\phi^t a, \phi^t b]$ and $g_t(K) = \frac{1}{\phi^t(b-a)}$ with $K \in [\phi^t a, \phi^t b]$. An illustration of the transformation over time is shown in the graph below:

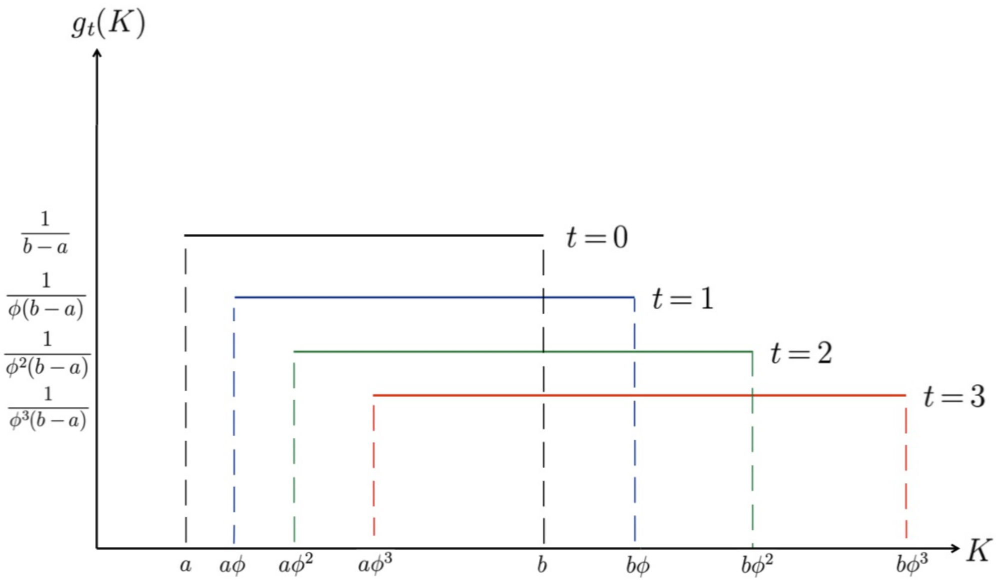

Figure 5. Growth of Capital - Evolvement of the Probability Density Function

Under the case of constant growth rate of capital, to keep the concept of higher entry barrier for manufacturing sector consistent, we impose that the exogenous entry cost c grows proportionally to firms' profitability, i.e. $c_t = c_0 \left(\frac{\phi^{2t\frac{\rho_g - 1}{\rho_g}} + 1}{\phi^{t\frac{\rho_g - 1}{\rho_g}} + 1} \right)^{\frac{\rho_g}{\rho_g - 1}}$.

Then, we derive the new equilibrium:

$$\begin{cases} \text{Wage: } w_{it}(L) = (A_{it})^{\frac{1}{\rho_i - 1}} P_{it} \, (L - a) + \mathcal{C} \cdot \mathbb{1}_{i=g} \\ \text{Profit: } \pi_{it}(K) = (A_{it})^{\frac{1}{\rho_i - 1}} P_{it} \left(\phi^{-\frac{t}{\rho_i}} K + a \right) - (\mathcal{C} + c_t) \cdot \mathbb{1}_{i=g} \\ \text{Price of service: } P_{st} = \frac{(A_{gt})^{\frac{\rho_g}{\rho_g - 1}}}{4\phi^t} [(b\phi^t)^2 - K_{*t}^2] \\ \text{Price of goods: } P_{gt} = \frac{(A_{st})^{\frac{\rho_s}{\rho_s - 1}}}{4\phi^t} [K_{*t}^2 - (a\phi^t)^2] \\ \text{ODE Constant: } \mathcal{C} = \left[(A_{st})^{\frac{1}{\rho_s - 1}} P_{st} - (A_{gt})^{\frac{1}{\rho_g - 1}} P_{gt} \right] \left(\frac{K_{*t}}{\phi^t} - a \right) \\ \text{Threshold Capital: } K_{*t} = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \end{cases}$$

where $L \in \left[a, \frac{K_{*t}}{\phi^t}\right]$ for $i = s$, $L \in \left(\frac{K_{*t}}{\phi^t}, b\right]$ for $i = g$ and:

$$\begin{cases} p = -\frac{N[(a\phi^t)^2 + (b\phi^t)^2]}{M + N} \\ q = -\frac{c_t}{M + N} \\ A_{it} = \phi^{\frac{\rho_i - 1}{\rho_i} \cdot t} + 1 \\ M = (A_{gt})^{\frac{1}{\rho_g - 1}} \frac{(A_{st})^{\frac{\rho_s}{\rho_s - 1}}}{4\phi^t} \left(\phi^{-\frac{t}{\rho_g}} + \frac{1}{\phi^t} \right) \\ N = (A_{st})^{\frac{1}{\rho_s - 1}} \frac{(A_{gt})^{\frac{\rho_g}{\rho_g - 1}}}{4\phi^t} \left(\phi^{-\frac{t}{\rho_s}} + \frac{1}{\phi^t} \right) \end{cases}$$

Proof. Appendix 3. □

3.2.1 Properties of the Equilibrium

Figure 6 shows the dynamics of profit function at time $0, 1, 2$. It is necessary to point out that time is continuous in the model. Here we are only sampling three moments in the continuous evolvement of profit, but the trend is already obvious - profit of all firms increase, and those with the highest capital are benedifted the most. Similar properties are shared by wages, as shown in figure 7.

Property. *(i). Firms that were in service sector are entering manufacturing sector, since the empoloyment share of service secotor* $\frac{L_* - a}{b - a}$ *decreases over time. Moreover, the employment share* ***stabilizes*** *as* $t \to \infty$*. (See figure 8)*

This property is an important result of type 1 automation. This shift in sector is due to an unneutral improvement in capital productivity. Firms with higher endowment K enjoys a greater increase in productivity, thus creating extra surplus for them to overcome the entry cost for manufacturing sector.

However, since the exogenous entry cost is also increasing with a speed of same order of profitability, as $t \to \infty$, service sector will be not be wiped out, but stablizes around 0.58. The aymptotic behavior gives us a stationary share of two sectors in the long term.

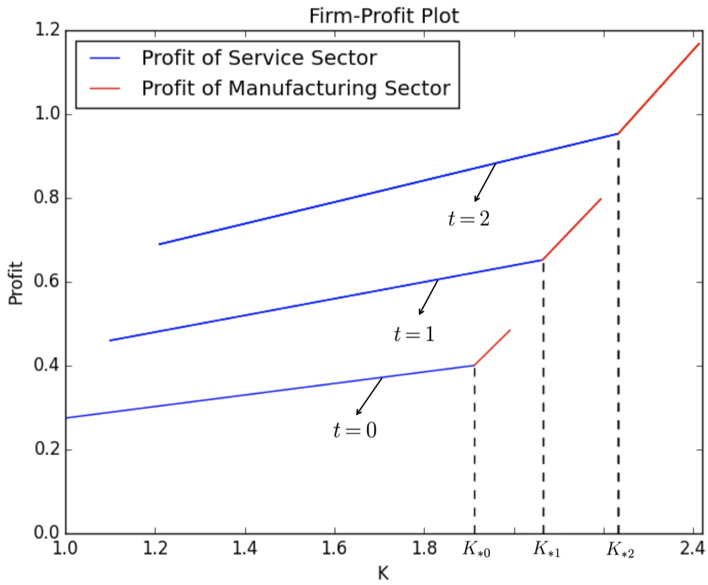

Figure 6. Evolvement of Profit Over Time

Property. *(ii). Even the worst firm enjoys an increase in profit, while the worst worker always earns zero wage.*

This property is a result of one-to-one matching. The worst worker has no bargaining power in negotiating his/her wage, and since there is no occupation choice, the worst worker has to accept a zero wage job offer. In this case, the worst firm is able to extract all the surplus and essentially produce with labor for free.

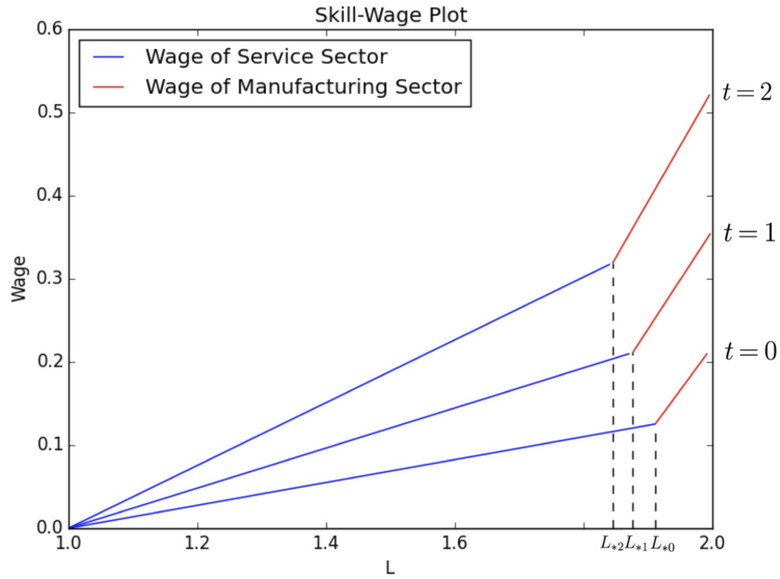

Figure 7. Evolvement of Wage Over Time

Property. *(iii). Profit and wage both increase in productivity of capital.*

This is a welcoming property of automation in that machines not only do not surpress human larbor, but also benefit us. The result is due to the complementarity between labor and capital in production - an increase in capital productivity also increases the marginal product of labor. However, profit increases faster than wage does, as a result of non-improving skill versus improving productivity of capital. It can be predicted that, over time, the proportion of human contribution to production will be very low.

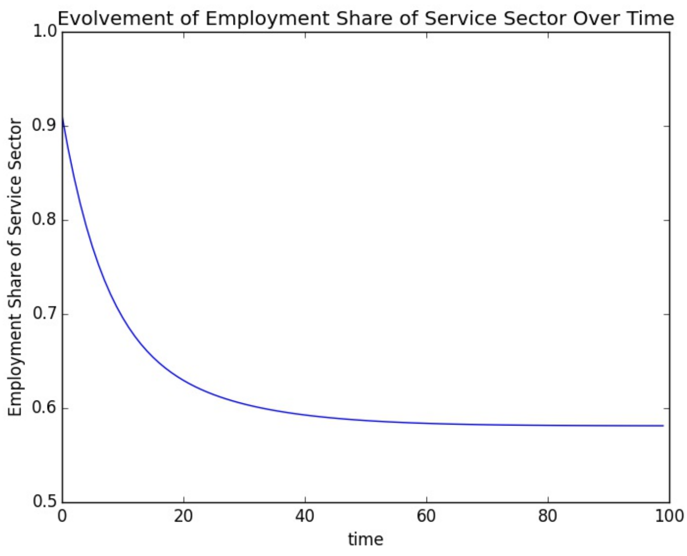

Figure 8. Evolvement of Employment Share of Service Sector Over Time

3.3 Type 2: Increasing Intensity of Capital in Production

In this type of automation, the previous simplified production function with fixed share parameter no longer works. Thus, we introduce the new production function, while keeping everything else unchanged:

Firms produce using the following CES technology:

$$y_i = \left(\beta K^{\frac{\rho_i - 1}{\rho_i}} + (1-\beta) L^{\frac{\rho_i - 1}{\rho_i}}\right)^{\frac{\rho_i}{\rho_i - 1}}, i \in \{s, g\} \tag{23}$$

Solve for equilibrium:

$$\begin{cases} \text{Wage: } w_i(L) = (1-\beta) P_i (L-a) + \mathcal{C} \cdot \mathbb{1}_{i=g} \\ \text{Profit: } \pi_i(K) = P_i [\beta K + (1-\beta) a] - (\mathcal{C} + c) \cdot \mathbb{1}_{i=g} \\ \text{Price of service: } P_s = \frac{b^2 - K_*^2}{4} \\ \text{Price of goods: } P_g = \frac{K_*^2 - a^2}{4} \\ \text{ODE Constant: } \mathcal{C} = (1-\beta)(P_s - P_g)(L_* - a) \\ \text{Threshold Capital: } K_* = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \end{cases}$$

where $p = -\frac{a^2 + b^2}{2}$, $q = -2c$, $L \in [a, K_*]$ for $i = s$ and $L \in (K_*, b]$ for $i = g$.

Proof. Appendix 4. □

3.3.1 Properties of the Equilibrium

Figure 9 and figure 10 show the dynamics of profit and wage with increases in β. Although profit is still increasing for all firms, wage decreases, which is the oppostie of type 1 automation. And decreasing wage is usually not an ideal type of automation.

Property. *(i). Profit increases in β, while wage decreases in β.*

Since capital is used more intensivity, the **relative** marginal product of capital increases compared to labor. And thus firms extract a greater share of surplus. Conversely, the relative marginal product of labor decreases, and workers extract a smaller share of surplus. This property is reminiscent of the estimation from Acemoglu and Restrepo (2017) that one more robot per thousand workers will reduce wages by 0.25-0.5 percent. Although here we do not have increasing stock of equipment, the sources of decrease in wage are similar - machines are more heavily relied on and, as a result, labor is valued less.

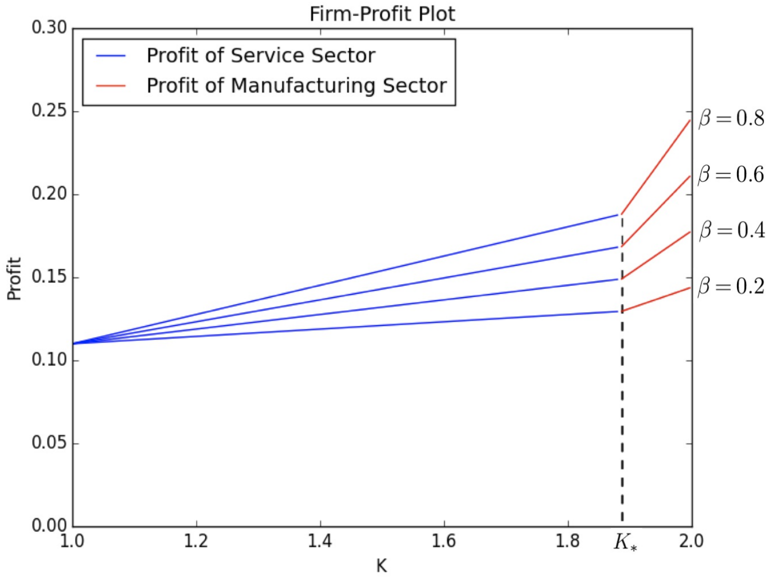

Figure 9. Evolvement of Profit Over Time

Figure 10. Evolvement of Wage Over Time

Property. *(ii). Employment share of each sector is constant.*

This property has very important implications on income distribution, which will be discussed below in section 4.1. Analytically, the expression of K_* is a constant independent of β. And to explain it through economics mechanism, this is because the effect of an increase in share parameter is neutral - both firms and workers are impacted proportionally across the entire population.

4 Income Inequality

4.1 Stationary Income Inequality for Type 2 Automation

Type 2 automation is the case with increasing intensity of capital in production. The following proposition is a result of the nature of changes in intensity (or share parameter).

Proposition 3. *Type 2 automation features a stationary income inequality, that is, income inequality neither gets worse nor gets improved. Thus, it has constant Gini coefficient and same Lorenz curve* $\forall \beta \in [0,1]$.

Proof. Recall that in equilibrium we have:

$$\begin{cases} \text{Wage: } w_i(L) = (1-\beta)P_i\,(L-a) + \mathcal{C}\cdot \mathbb{1}_{i=g} \\ \text{Price of service: } P_s = \frac{b^2 - K_*^2}{4} \qquad \text{Price of goods: } P_g = \frac{K_*^2 - a^2}{4} \\ \text{Threshold Capital: } K_* = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \end{cases}$$

where $p = -\frac{a^2+b^2}{2}$, $q = -2c$, $L \in [a, K_*]$ for $i = s$ and $L \in (K_*, b]$ for $i = g$.

It is worth noting that K_* is constant given fixed upper and lower bound of labor skills a and b.

$\Rightarrow P_s$ and P_g are also constants.

$\Rightarrow$Slope of wage functions w_s and w_g are $(1-\beta)P_s$ and $(1-\beta)P_g$. They are functions of β only.

Since the change in β is global to all firms, all workers face the same change in wage proportionally. Given that the distribution of labor is stationary, although the absolute amount of wage increases, worker's wage relative to others stays the same. □

4.2 Stationary Gini Coefficient for Type 1 in Long Term

As can be seen from wage plots (figure 7 and figure 10), in both types of automation, the wealthier workers are getting a greater increase in wage. This makes the resulting income distribution of interest to us. Since both types of automation feature wage functions that are linear in L, it is easy to come up with the Gini coefficient of each. The general expression is:

$$\text{Gini} = \frac{L_* - a}{b - a} - \frac{(L_* - a)w(L_*)}{(b-a)w(L_*) + (b - L_*)w(b)}$$

The derivation is shown in Appendix 5.

It can be seen from figure 11 that the Gini coefficient is increasing at the beginning, then it decreases and approaches an asymptotic line around 0.174. The asymptotic behavior is a result from the asymptotic employment share of two sectors discussed in section 3.2.1.

As a comparison, Iceland, of which the Gini coefficient is 0.24 in 2012, is the lowest among OECD countries. Thus, the economy under type 1 automation is much more equal than the most equal country nowadays. Intuitively, from the piecewise linear wage, we can already have a sense of why the economy in the model has such low Gini coefficient, compared to the usually quadratic shaped curves in real life.

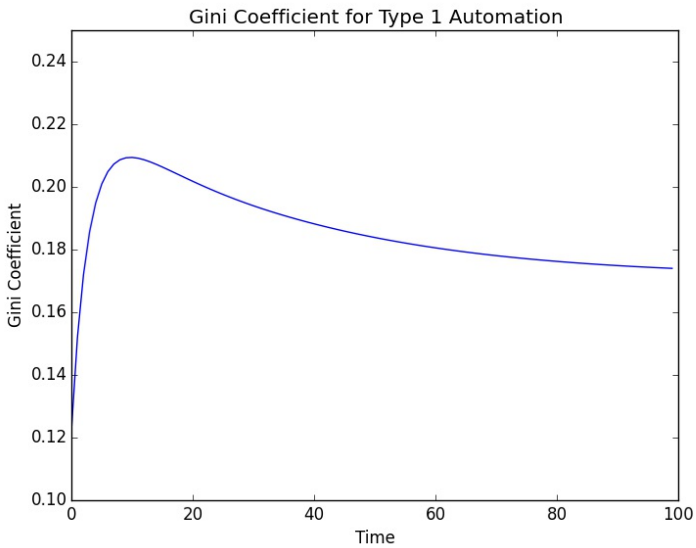

Figure 11. Gini coefficient for Type 1 Automation

4.3 Percentage Wealth Owned by Top 1%, 5%, 20% and Bottom 50% Workers and Lorenz Curve

As shown in section 4.1, type 2 automation features a stationary income distribution, so we will only discuss the case of type 1 automation in this section.

Figure 12 shows the percentage of wealth owned by top 1%, 5% and 20% workers. It could be striking at first glance to find that their shares are decreasing over time. But it will all start to make sense if we look at the wage curve in figure 7 - the percentage of workers in manufacturing sector, the high wage sector, is growing. And as more workers earning high wage in manufacturing sector (which is linear within the same sector), the premium earned by the very few top workers at the beginning starts to shrink.

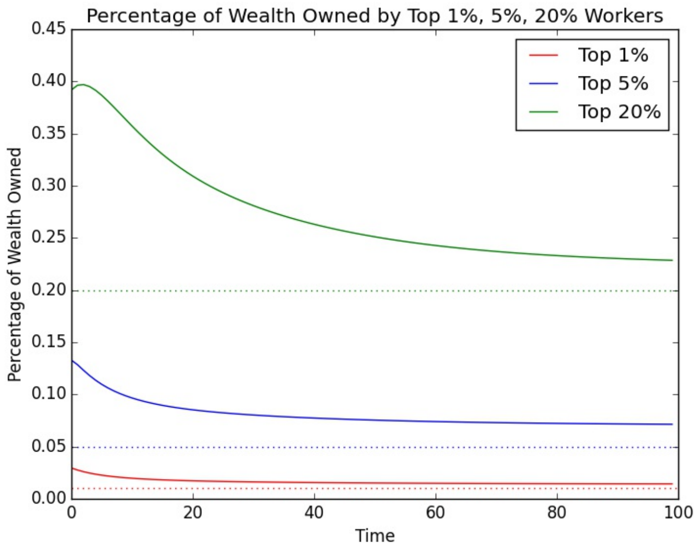

Figure 12. Percentage Wealth Owned by Top 1%, 5% and 20% Workers

And it is very understandable that as the percentage wealth owned by top workers start to take a dip, that owned by the bottom workers will rise. Since inequality still exists, the bottom 50% workers, as shown in figure 13, increases and approaches an asymptotic line around 0.30. The dynamics between top and bottom workers' wealth share is better illustrated in the Lorenz curve in figure 14.

Figure 13. Percentage Wealth Owned by Bottom 50% Workers

As in figure 14, the kink is where the separation of service sector and manufacturing sector occurs. As discussed in section 3.2.1, the employment share stabilizes in the long run, thus, Lorenz curve also has asymptotic behavior in the long run, which is close to the curve at time = 100.

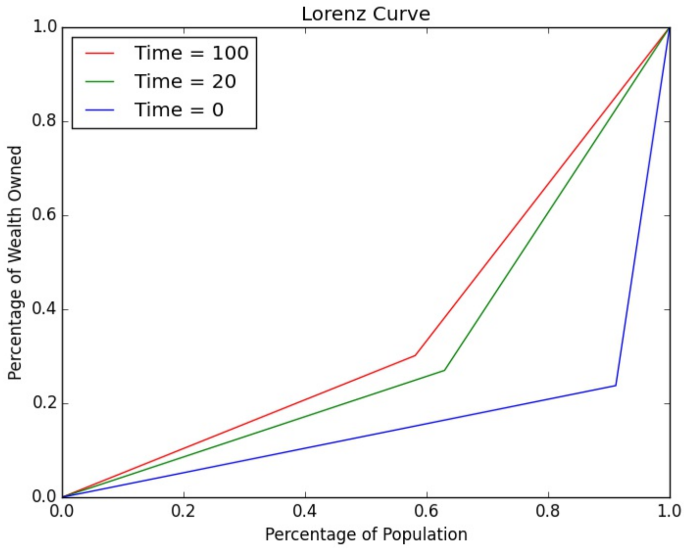

Figure 14. Lorenz Curve

5 Conclusion

This paper proposes two types of automation, in which we do not require the creation of new tasks to keep labor from being redundant. Type 1 automation features a non-trivial stationary employment share and income inequality in the long run. Thus, type 1 automation, i.e. the increases in productivity of capital, is the friendlier form of automation to human labor, since labor as a complementary production factor to capital benefits from the increased productivity of capital with its own marginal product increasing, incidentally. The stationary income inequality among workers in the long run especially makes this type of automation attractive.

Type 2 automation that uses capital more intensively relative to labor reduces the need for human labor over time in the form of decreased wage, which is essentially transfered into firms' profit. With workers being suppressed, this type of automation is not in favor of workers, although it features constant employment share and income inequality at all time.

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Appendix

1 Solving for $\mathbb{K}_*$ in the Baseline Model

Equations (15)(16)(17)(18) are: (we label (15)(16)(17)(18) as (1)(2)(3)(4) below to simplify notation.)

$$2^{\frac{1}{\rho_s-1}} P_s (K_* - a) = 2^{\frac{1}{\rho_g-1}} P_g (K_* - a) + \mathcal{C} \qquad (1)$$

$$2^{\frac{1}{\rho_s-1}} P_s (K_* + a) = 2^{\frac{1}{\rho_g-1}} P_g (K_* + a) - \mathcal{C} - c \qquad (2)$$

$$P_s = 2^{\frac{2-\rho_g}{\rho_g-1}} (b^2 - K_*^2) \qquad (3)$$

$$P_g = 2^{\frac{2-\rho_s}{\rho_s-1}} (K_*^2 - a^2) \qquad (4)$$

From (1):

$$\Rightarrow \quad -\mathcal{C} = 2^{\frac{1}{\rho_g-1}} P_g (K_* - a) - 2^{\frac{1}{\rho_s-1}} P_s (K_* - a) \quad (5)$$

Plug (5) into (2):

$$\left(2^{\frac{\rho_s}{\rho_s-1}} - 2^{\frac{1}{\rho_s-1}}\right) P_s (K_* + a) = \left(2^{\frac{\rho_g}{\rho_g-1}} - 2^{\frac{1}{\rho_g-1}}\right) P_g (K_* + a) - \mathcal{C} - c$$

$$\Rightarrow \qquad c = 2^{\frac{\rho_g+\rho_s-2}{(\rho_g-1)(\rho_s-1)}} [2K_*^3 - (a^2+b^2)K_*]$$

Solve the zeros of the depressed cubic function:

$$\Rightarrow \quad K_*^3 - \frac{a^2+b^2}{2} K_* - c \cdot 2^{\frac{1-\rho_g\rho_s}{(\rho_g-1)(\rho_s-1)}} = 0$$

From *Cardano's method*, the real root is:

$$K_* = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

where $p = -\frac{a^2+b^2}{2}$ and $q = -c \cdot 2^{\frac{1-\rho_g\rho_s}{(\rho_g-1)(\rho_s-1)}}$.

2 Derivation of Equation (22) - Distribution of Capital under Constant Growth Rate

At $t=0$, $G_0(K)\sim U[a,b]$, since we assume a growth rate of capital such that $K_{t+1}=\phi K_t$, we have:

$$G_{t+1}(K)=\Pr(K_{t+1}\leqslant K)=\Pr(\phi K_t\leqslant K)=G_t\left(\frac{K}{\phi}\right)$$

Therefore,

$$G_t(K)=G_{t-1}\left(\frac{K}{\phi}\right)=\cdots=G_0\left(\frac{K}{\phi^t}\right)=\frac{K-\phi^t a}{\phi^t(b-a)}, K\in[\phi^t a,\phi^t b]$$

and

$$g_t(K)=\frac{1}{\phi^t}g_0\left(\frac{K}{\phi^t}\right)=\frac{1}{\phi^t(b-a)}, K\in[\phi^t a,\phi^t b]$$

3 Solving for Equilibrium in Type 1 Automation

Equilibrium conditions:

Recall that $L\sim U[a,b]$ and $G_t(K)\sim[\phi^t a,\phi^t b]$.

***1. Labor Market Clearing Conditions*:**

$$1-F(\alpha(K))=1-G_t(K) \tag{24}$$

Since $L\sim U[a,b]$ and $G_t(K)\sim[\phi^t a,\phi^t b]$, equation (24) gives:

$$L=\alpha(K)=\frac{K}{\phi^t}, K\in[\phi^t a,\phi^t b] \tag{25}$$

***2. Profit Maximization*:**

$$\underset{L}{\operatorname{Max}} \quad P_{it}\left(K_t^{\frac{\rho_i-1}{\rho_i}}+L_t^{\frac{\rho_i-1}{\rho_i}}\right)^{\frac{\rho_i}{\rho_i-1}}-w_{it}(L_t)-c_t\cdot\mathbb{1}_{i=g} \tag{26}$$

where $i\in\{s,g\}$ and c is the exogenous constant entry cost for firms in the manufacturing sector.

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$$P_{it}\left(K_t^{\frac{\rho_i-1}{\rho_i}}+L_t^{\frac{\rho_i-1}{\rho_i}}\right)^{\frac{1}{\rho_i-1}}L_t^{-\frac{1}{\rho_i}}=\frac{d}{d\,L_t}w_{it}(L_t) \tag{27}$$

$$\Rightarrow \qquad \left(\phi^{\frac{\rho_i-1}{\rho_i}\cdot t}+1\right)^{\frac{1}{\rho_i-1}}P_{it}=\frac{d}{d\,L_t}w_{it}(L_t) \tag{28}$$

Write $\left(\phi^{\frac{\rho_i-1}{\rho_i}\cdot t}+1\right)=A_{it}$. Then solve the ODE in equation (11) with initial condition $w_s(a)=0$, while we do not restrict the initial condition for w_g:

$$w_{it}(L)=\begin{cases}(A_{st})^{\frac{1}{\rho_s-1}}P_{st}\,(L-a) & \text{for } i=s \text{ and } L\in[a,L_*]\\ (A_{gt})^{\frac{1}{\rho_g-1}}P_{gt}\,(L-a)+\mathcal{C} & \text{for } i=g \text{ and } L\in(L_*,b]\end{cases} \tag{29}$$

where $\mathcal{C}$ is an arbitrary constant to be determined endogenously, $i\in\{s,g\}$, $L\in[0, L_*]$ for $i=s$ and $L\in(L_*,1\}$ for $i=g$. $A_{it}=\phi^{\frac{\rho_i-1}{\rho_i}\cdot t}+1$.

Equation (29) can also be written as:

$$w_{it}(L) = (A_{it})^{\frac{1}{\rho_i - 1}} P_{it}\,(L - a) + \mathcal{C} \cdot \mathbb{1}_{i=g}$$

***3. Firms' Optimal Choice of Sector*:**

In equilibrium, firms optimize their choices of sectors so that it is not profitable to switch to the other sector. From equation (25) and with equilibrium wage (29), we can write equation (26), firms' profit, as:

$$\pi_{it}(K) = (A_{it})^{\frac{1}{\rho_i - 1}} P_{it} \left(\phi^{-\frac{t}{\rho_i}} K + a \right) - (\mathcal{C} + c_t) \cdot \mathbb{1}_{i=g} \tag{30}$$

which is linear in K under equilibrium. Therefore, the intersection of two sectors' profit function will determine the threshold capital level K_{*t}, which satisfies:

$$(A_{st})^{\frac{1}{\rho_s - 1}} P_{st} \left(\phi^{-\frac{t}{\rho_s}} K_{*t} + a \right) = (A_{gt})^{\frac{1}{\rho_g - 1}} P_{gt} \left(\phi^{-\frac{t}{\rho_g}} K_{*t} + a \right) - (\mathcal{C} + c_t) \tag{31}$$

4. Workers' Indifference Condition:

With mobility between sectors, we need to impose an indifference condition such that the marginal agent $L_{*t} = \frac{K_{*t}}{\phi^t}$ is indifferent between working in both sectors:

$$\text{then} \quad \lim_{L \uparrow L_*^-} w_{st}(L) = \lim_{L \downarrow L_*^+} w_{gt}(L)$$

Thus, this gives us the initial condition for $w_g(L)$ such that $w_g(L_*) = w_s(L_*) = (A_{st})^{\frac{1}{\rho_s - 1}} P_{st}\, L_*$,

$$\Rightarrow \quad (A_{st})^{\frac{1}{\rho_s - 1}} P_{st}\,(L_{*t} - a) = (A_{gt})^{\frac{1}{\rho_g - 1}} P_{gt}\,(L_{*t} - a) + \mathcal{C} \tag{32}$$

From equation (5), we also know that:

$$P_{st} = \frac{1}{2} Y_{gt} = \frac{1}{2} \int_{K_{*t}}^{b\phi^t} \left(K_t^{\frac{\rho_g - 1}{\rho_g}} + L^{\frac{\rho_g - 1}{\rho_g}} \right)^{\frac{\rho_g}{\rho_g - 1}} \mathrm{dK} = \frac{(A_{gt})^{\frac{\rho_g}{\rho_g - 1}}}{4\phi^t} [(b\phi^t)^2 - K_{*t}^2] \tag{33}$$

$$P_{gt} = \frac{1}{2} Y_{st} = \frac{1}{2} \int_{a\phi^t}^{K_{*t}} \left(K_t^{\frac{\rho_s - 1}{\rho_s}} + L^{\frac{\rho_s - 1}{\rho_s}} \right)^{\frac{\rho_s}{\rho_s - 1}} \mathrm{dK} = \frac{(A_{st})^{\frac{\rho_s}{\rho_s - 1}}}{4\phi^t} [K_{*t}^2 - (a\phi^t)^2] \tag{34}$$

Combine equation (31)(32)(33)(34), we can solve for P_{st}, P_{gt}, $\mathcal{C}$ and K_{*t}: (we label (31)(32)(33)(34) as (1)(2)(3)(4) below to simplify notation.)

$$(A_{st})^{\frac{1}{\rho_s - 1}} P_{st}\,(L_{*t} - a) = (A_{gt})^{\frac{1}{\rho_g - 1}} P_{gt}\,(L_{*t} - a) + \mathcal{C} \tag{1}$$

$$(A_{st})^{\frac{1}{\rho_s - 1}} P_{st} \left(\phi^{-\frac{t}{\rho_s}} K_{*t} + a \right) = (A_{gt})^{\frac{1}{\rho_g - 1}} P_{gt} \left(\phi^{-\frac{t}{\rho_g}} K_{*t} + a \right) - (\mathcal{C} + c_t) \tag{2}$$

$$P_{st} = \frac{(A_{gt})^{\frac{\rho_g}{\rho_g - 1}}}{4\phi^t} [(b\phi^t)^2 - K_{*t}^2] \tag{3}$$

$$P_{gt} = \frac{(A_{st})^{\frac{\rho_s}{\rho_s - 1}}}{4\phi^t} [K_{*t}^2 - (a\phi^t)^2] \tag{4}$$

From (1):

$$\Rightarrow \quad -\mathcal{C} = (A_{gt})^{\frac{1}{\rho_g - 1}} P_{gt} \left(\frac{K_{*t}}{\phi^t} - a \right) - (A_{st})^{\frac{1}{\rho_s - 1}} P_{st} \left(\frac{K_{*t}}{\phi^t} - a \right) \tag{5}$$

Plug (5) into (2):

$$(A_{st})^{\frac{1}{\rho_s-1}} P_{st}\left(\phi^{-\frac{t}{\rho_s}} K_{*t}+a\right) = (A_{gt})^{\frac{1}{\rho_g-1}} P_{gt}\left(\phi^{-\frac{t}{\rho_g}} K_{*t}+a\right) - \mathcal{C} - c_t$$

$$\Rightarrow \quad c_t = \left[(A_{gt})^{\frac{1}{\rho_g-1}} P_{gt}\left(\phi^{-\frac{t}{\rho_g}}+\frac{1}{\phi^t}\right) - (A_{st})^{\frac{1}{\rho_s-1}} P_{st}\left(\phi^{-\frac{t}{\rho_s}}+\frac{1}{\phi^t}\right)\right] K_{*t}$$

$$\left[(A_{gt})^{\frac{1}{\rho_g-1}} \frac{(A_{st})^{\frac{\rho_s}{\rho_s-1}}}{4\phi^t}\left(\phi^{-\frac{t}{\rho_g}}+\frac{1}{\phi^t}\right)[K_{*t}^2-(a\phi^t)^2] - (A_{st})^{\frac{1}{\rho_s-1}} \frac{(A_{gt})^{\frac{\rho_g}{\rho_g-1}}}{4\phi^t}\left(\phi^{-\frac{t}{\rho_s}}+\frac{1}{\phi^t}\right)[(b\phi^t)^2-K_{*t}^2]\right] K_{*t}$$

$$c_t = \left[(A_{gt})^{\frac{1}{\rho_g-1}} \frac{(A_{st})^{\frac{\rho_s}{\rho_s-1}}}{4\phi^t}\left(\phi^{-\frac{t}{\rho_g}}+\frac{1}{\phi^t}\right) + (A_{st})^{\frac{1}{\rho_s-1}} \frac{(A_{gt})^{\frac{\rho_g}{\rho_g-1}}}{4\phi^t}\left(\phi^{-\frac{t}{\rho_s}}+\frac{1}{\phi^t}\right)\right] K_{*t}^3$$

$$-(A_{st})^{\frac{1}{\rho_s-1}} \frac{(A_{gt})^{\frac{\rho_g}{\rho_g-1}}}{4\phi^t}\left(\phi^{-\frac{t}{\rho_s}}+\frac{1}{\phi^t}\right)[(a\phi^t)^2+(b\phi^t)^2]\, K_{*t}$$

Denote $(A_{gt})^{\frac{1}{\rho_g-1}} \frac{(A_{st})^{\frac{\rho_s}{\rho_s-1}}}{4\phi^t}\left(\phi^{-\frac{t}{\rho_g}}+\frac{1}{\phi^t}\right)$ as M and $(A_{st})^{\frac{1}{\rho_s-1}} \frac{(A_{gt})^{\frac{\rho_g}{\rho_g-1}}}{4\phi^t}\left(\phi^{-\frac{t}{\rho_s}}+\frac{1}{\phi^t}\right)$ as N, then solve the zeros of the depressed cubic function:

$$(M+N)K_{*t}^3 - N[(a\phi^t)^2+(b\phi^t)^2]K_{*t} - c_t = 0$$

$$\Rightarrow \quad K_{*t}^3 - \frac{N[(a\phi^t)^2+(b\phi^t)^2]}{M+N} K_{*t} - \frac{c_t}{M+N} = 0$$

From *Cardano's method*, the real root is:

$$K_{*t} = \sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}$$

where $p = -\frac{N[(a\phi^t)^2+(b\phi^t)^2]}{M+N}$ and $q = -\frac{c_t}{M+N}$.

Combine equations (31)(32)(33)(34), we can solve for $P_{st}, P_{gt}, \mathcal{C}$ and K_{*t}:

$$\begin{cases}
\text{Wage: } w_{it}(L) = (A_{it})^{\frac{1}{\rho_i-1}} P_{it}\,(L-a) + \mathcal{C}\cdot \mathbb{1}_{i=g} \\
\text{Profit: } \pi_{it}(K) = (A_{it})^{\frac{1}{\rho_i-1}} P_{it}\left(\phi^{-\frac{t}{\rho_i}} K + a\right) - (\mathcal{C}+c_t)\cdot \mathbb{1}_{i=g} \\
\text{Price of service: } P_{st} = \frac{(A_{gt})^{\frac{\rho_g}{\rho_g-1}}}{4\phi^t}[(b\phi^t)^2 - K_{*t}^2] \\
\text{Price of goods: } P_{gt} = \frac{(A_{st})^{\frac{\rho_s}{\rho_s-1}}}{4\phi^t}[K_{*t}^2 - (a\phi^t)^2] \\
\text{ODE Constant: } \mathcal{C} = \left[(A_{st})^{\frac{1}{\rho_s-1}} P_{st} - (A_{gt})^{\frac{1}{\rho_g-1}} P_{gt}\right]\left(\frac{K_{*t}}{\phi^t} - a\right) \\
\text{Threshold Capital: } K_{*t} = \sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}
\end{cases}$$

where $L \in \left[a, \frac{K_{*t}}{\phi^t}\right]$ for $i = s$ and $L \in \left(\frac{K_{*t}}{\phi^t}, b\right]$ for $i = g$. And:

$$\begin{cases} p = -\frac{N[(a\phi^t)^2 + (b\phi^t)^2]}{M+N} \\ q = -\frac{c_t}{M+N} \\ A_{it} = \phi^{\frac{\rho_i - 1}{\rho_i} \cdot t} + 1 \\ M = (A_{gt})^{\frac{1}{\rho_g - 1}} \frac{(A_{st})^{\frac{\rho_s}{\rho_s - 1}}}{4\phi^t} \left(\phi^{-\frac{t}{\rho_g}} + \frac{1}{\phi^t}\right) \\ N = (A_{st})^{\frac{1}{\rho_s - 1}} \frac{(A_{gt})^{\frac{\rho_g}{\rho_g - 1}}}{4\phi^t} \left(\phi^{-\frac{t}{\rho_s}} + \frac{1}{\phi^t}\right) \end{cases}$$

4 Solving for Equilibrium in Type 0 and Type 2 Automation

Equilibrium conditions:

***1. Labor Market Clearing Conditions*:**

Same as before:

$$1 - F(\alpha(K)) = 1 - G(K) \tag{35}$$

Since $F \sim U[a, b]$ and $G \sim U[a, b]$, equation (35) gives:

$$L = \alpha(K) = K \tag{36}$$

Therefore, $K_* = L_*$, i.e. the marginal worker L_* between two sectors has skill level K_* and, in addition, workers with $L \in [a, L_*]$ are employed by the service sector while those with $L \in (L_*, b]$ are employed by the manufacturing sector.

***2. Profit Maximization*:**

Firms maximize profit and we denote wage for service sector as w_s, and w_g for manufacturing sector:

$$\underset{L}{\text{Max}} \quad P_i \left(\beta K^{\frac{\rho_i - 1}{\rho_i}} + (1-\beta) L^{\frac{\rho_i - 1}{\rho_i}}\right)^{\frac{\rho_i}{\rho_i - 1}} - w_i(L) - c \cdot \mathbb{1}_{i=g} \tag{37}$$

where $i \in \{s, g\}$ and c is the exogenous constant entry cost for firms in the manufacturing sector.

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$$(1-\beta) P_i \left(\beta K^{\frac{\rho_i - 1}{\rho_i}} + (1-\beta) L^{\frac{\rho_i - 1}{\rho_i}}\right)^{\frac{1}{\rho_i - 1}} L^{-\frac{1}{\rho_i}} = \frac{d}{dL} w_i(L) \tag{38}$$

$$\Rightarrow \qquad (1-\beta)\, P_i = \frac{d}{dL} w_i(L) \tag{39}$$

Solve the ODE in equation (39) with initial condition $w_s(a) = 0$, while we do not restrict the initial condition for w_g:

$$w(L) = \begin{cases} (1-\beta) P_s (L - a) & \text{for } i = s \text{ and } L \in [a, L_*] \\ (1-\beta) P_g (L - a) + \mathcal{C} & \text{for } i = g \text{ and } L \in (L_*, b] \end{cases} \tag{40}$$

where $\mathcal{C}$ is an arbitrary constant to be determined endogenously, $i\in\{s,g\}$, $L\in[a,L_*]$ for $i=s$ and $L\in(L_*,b]$ for $i=g$.

Equation (40) can be rewritten as:

$$w_i(L)=(1-\beta)P_i\,(L-a)+\mathcal{C}\cdot\mathbb{1}_{i=g} \tag{41}$$

***3. Firms' Optimal Choice of Sector*:**

In equilibrium, firms optimize their choices of sectors so that it is not profitable to switch to the other sector. From equation (36) and with equilibrium wage (13), we can write equation (37), firms' profit, as:

$$\pi_i(K)=P_i\,[\beta\,K+(1-\beta)a]-(\mathcal{C}+c)\cdot\mathbb{1}_{i=g} \tag{42}$$

which is linear in K under equilibrium. Therefore, the intersection of two sectors' profit function will determine the threshold capital level K_*, which satisfies:

$$P_s\,[\beta\,K_*+(1-\beta)a]=P_g\,[\beta\,K_*+(1-\beta)a]-(\mathcal{C}+c) \tag{43}$$

4. *Workers' Indifference Condition:*

With mobility between sectors, we need to impose an indifference condition such that the marginal agent $L_*=K_*$ is indifferent between working in both sectors:

$$\text{then}\quad \lim_{L\uparrow L_*^-} w_s(L)=\lim_{L\downarrow L_*^+} w_g(L)$$

Thus, this gives us the initial condition for $w_g(L)$ such that $w_g(L_*)=w_s(L_*)=2^{\frac{1}{\rho_s-1}}P_s\,L_*$,

$$\Rightarrow\;\;(1-\beta)P_s\,(L_*-a)=(1-\beta)P_g\,(L_*-a)+\mathcal{C} \tag{44}$$

From equation (5), we also know that:

$$P_s=\frac{1}{2}Y_g=\frac{1}{2}\int_{K_*}^{b}\Big(\beta K^{\frac{\rho_g-1}{\rho_g}}+(1-\beta)L^{\frac{\rho_g-1}{\rho_g}}\Big)^{\frac{\rho_g}{\rho_g-1}}\,\mathrm{dK}=\frac{b^2-K_*^2}{4} \tag{45}$$

$$P_g=\frac{1}{2}Y_s=\frac{1}{2}\int_{a}^{K_*}\Big(\beta K^{\frac{\rho_s-1}{\rho_s}}+(1-\beta)L^{\frac{\rho_s-1}{\rho_s}}\Big)^{\frac{\rho_s}{\rho_s-1}}\,\mathrm{dK}=\frac{K_*^2-a^2}{4} \tag{46}$$

Combine equations (43)(44)(45)(46), we can solve for P_{st}, P_{gt}, $\mathcal{C}$ and K_{*t}: (we label (43)(44)(45)(46) as (1)(2)(3)(4) below to simplify notation.)

$$\begin{aligned}(1-\beta)P_s\,(L_*-a)&=(1-\beta)P_g\,(L_*-a)+\mathcal{C} && (1)\\ P_s\,[\beta\,K+(1-\beta)a]&=P_g\,[\beta\,K+(1-\beta)a]-(\mathcal{C}+c) && (2)\\ P_{st}&=\frac{b^2-K_*^2}{4} && (3)\\ P_{gt}&=\frac{K_*^2-a^2}{4} && (4)\end{aligned}$$

From (1):

$$\Rightarrow\;\;-\mathcal{C}=(1-\beta)(P_g-P_s)(L_*-a)\quad(5)$$

Plug (5) into (2):

$$\begin{aligned} P_s\left[\beta K + (1-\beta)a\right] &= P_g\left[\beta K + (1-\beta)a\right] - \mathcal{C} - c \\ c &= (P_g - P_s)\, K_* \\ \Rightarrow \qquad K_* &= \frac{4c}{2K_*^2 - a^2 - b^2} \end{aligned}$$

Therefore, we get a cubic equation of K_*:

$$K_*^3 - \frac{a^2+b^2}{2} K_* - 2c = 0$$

From *Cardano's method*, the real root is:

$$K_* = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

where $p = -\frac{a^2+b^2}{2}$ and $q = -2c$.

$$\begin{cases} \text{Wage: } w_i(L) = (1-\beta)P_i\,(L-a) + \mathcal{C}\cdot \mathbb{1}_{i=g} \\ \text{Profit: } \pi_i(K) = P_i\left[\beta K + (1-\beta)a\right] - (\mathcal{C} + c)\cdot \mathbb{1}_{i=g} \\ \text{Price of service: } P_s = \frac{b^2 - K_*^2}{4} \\ \text{Price of goods: } P_g = \frac{K_*^2 - a^2}{4} \\ \text{ODE Constant: } \mathcal{C} = (1-\beta)(P_s - P_g)(K_* - a) \\ \text{Threshold Capital: } K_* = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \end{cases}$$

where $p = -\frac{a^2+b^2}{2}$, $q = -2c$, $L \in [a, K_*]$ for $i = s$ and $L \in (K_*, b]$ for $i = g$.

5 Derivation of Gini Coefficient

Aggregate wage earned by workers belonging to $[a, L_*]$ is:

$$\mathrm{AW}_s = \frac{(L_* - a) w_g(L_*)}{2}$$

Aggregate wage earned by worker belonging to $[L_*, b]$ is:

$$\mathrm{AW}_g = \frac{[w_g(L_*) + w_g(b)](b - L_*)}{2}$$

Total wealth is:

$$\mathrm{TW} = \frac{(L_* - a) w_g(L_*) + [w_g(L_*) + w_g(b)](b - L_*)}{2}$$

Proportion of wealth owned by workers belonging to $[a, L_*]$ is:

$$\mathrm{Prop}_s = \frac{\mathrm{AW}_s}{\mathrm{TW}}$$

Proportion of workers belonging to $[a, L_*]$ is:

$$h = \frac{L_* - a}{b - a}$$

Gini Coefficient is:

$$\begin{aligned} 1 - (h\,\mathrm{Prop}_s + (\mathrm{Prop}_s + 1)(1 - h)) \\ = h - \mathrm{Prop}_s \\ = \frac{L_* - a}{b - a} - \frac{(L_* - a)w_g(L_*)}{(L_* - a)w_g(L_*) + [w_g(L_*) + w_g(b)](b - L_*)} \end{aligned}$$